Antisymmetric elements in group rings with an orientation morphism

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Abstract

Let R be a commutative ring, G a group and RG its group ring. Let $\varphi_\sigma : RG \to RG$ denote the involution defined by $\varphi_\sigma(\sum r_g g) = \sum r_g \sigma(g) g^{-1}$, where $\sigma : G \to \{\pm 1\}$ is a group homomorphism (called an orientation morphism). An element x in RG is said to be antisymmetric if $\varphi_\sigma(x) = -x$. We give a full characterization of the groups G and its orientations for which the antisymmetric elements of RG commute.

1 Introduction

Let R be a commutative ring with unity and let G be a group. Let φ be an involution on the group ring RG. Denote by $\mathcal{U}(RG)$ the group of units of the group ring RG and by $(RG)^-_\varphi$ the set of its antisymmetric elements, that is,

$$(RG)^-_\varphi = \{\alpha \in RG \mid \varphi(\alpha) = -\alpha\}.$$

In this paper we investigate when $(RG)^-_\varphi$ is commutative, that is $ab = ba$ for all $a, b \in (RG)^-_\varphi$.

The group of φ-unitary units of RG is defined by

$$\mathcal{U}_\varphi(RG) = \{u \in \mathcal{U}(RG) \mid u\varphi(u) = 1\}.$$

For general algebras there is a close relationship between the φ-unitary units and the antisymmetric elements. For example, in [7] Giambruno and Polcino Milies show that if φ is an involution on a finite dimensional semisimple algebra A over an algebraically closed field F with $char(F) \neq 2$ then $\mathcal{U}_\varphi(A)$ satisfies a group identity if and only if $(A)^-_\varphi$ is commutative. Moreover, if F is a nonabsolute field then $\mathcal{U}_\varphi(A)$ does not contain a free group of rank 2 if and only if $(A)^-_\varphi$ is commutative. Giambruno and Sehgal, in [8], showed that if B is a semiprime ring with involution φ, $B = 2B$ and $(B)^-_\varphi$ is Lie nilpotent then $(B)^-_\varphi$ is commutative and B satisfies a polynomial identity of degree 4. The latter shows that crucial information of the algebraic structure of A can be determined by that of $(A)^-_\varphi$. We state two more important results of this nature. Amitsur in [1] proves that for an arbitrary algebra A with an involution φ, if A^-_φ satisfies a polynomial identity (in particular when A^-_φ is commutative) then A satisfies a polynomial identity. Gupta and Levin in [11] proved that for all $n \geq 1$ $\gamma_n(\mathcal{U}(A)) \leq 1 + L_n(A)$. Here $\gamma_n(G)$ denotes the nth term in the lower central series of the group G and $L_n(A)$ denotes the two sided ideal of A generated by all Lie elements

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of the form $[a_1, a_2, \ldots, a_n]$ with $a_i \in A$ and $[a_1] = a_1$, $[a_1, a_2] = a_1a_2 - a_2a_1$ and inductively $[a_1, a_2, \ldots, a_n] = [[a_1, a_2, \ldots, a_{n-1}], a_n]$. Smirnov and Zalesskii in [17], proved that, for example, if the Lie ring generated by the elements of the form $g + g^{-1}$ with $g \in \mathcal{U}(A)$ is Lie nilpotent then A is Lie nilpotent.

Special attention has been given to the classical involution $*$ on RG that is the R-linear map defined by mapping $g \in G$ onto g^{-1}. In case R is a field of characteristic 0 and G is a periodic group, Giambruno and Polcino Milies in [7] described when $\mathcal{U}_*(RG)$ satisfies a group identity. Gonçalves and Passman in [10] characterized when $\mathcal{U}_*(RG)$ does not contain non abelian free groups when G is a finite group and R is a nonabsolute field. Giambruno and Sehgal, in [8], show that if R is a field of characteristic $p \geq 0$, with $p \neq 2$ and G a group without 2-elements, then the Lie nilpotence of $(RG)^-_*$ implies the Lie nilpotence of RG. Giambruno, Polcino Milies and Sehgal in [7, 9] characterized when $(RG)^-_*$ is Lie nilpotent.

Because of all the above mentioned results, it is relevant to determine when the antisymmetric elements of a group ring commute. Recently, for an arbitrary involution φ on a group G (extended by linearity to RG) and a commutative ring R, Jespers and Ruiz [13] characterized when $(RG)^-_\varphi$ is commutative. This generalizes earlier work of Broche and Polcino Milies [2] in case φ is the classical involution. The characterizations obtained in both papers are in terms of the algebraic structure of some subgroups of G.

In [3], [4], [5] and [14] various authors considered involutions on a group ring RG that are not determined by R-linearity by an involution on G. The following is an example of such an involution φ_σ that was introduced by Novikov in [15] in the context of K-theory and algebraic topology:

$$\varphi_\sigma\left(\sum_{g\in G}\alpha_g g\right) = \sum_{g\in G}\alpha_g\sigma(g)g^{-1},$$

where $\sigma : G \to \{\pm 1\}$ is a group homomorphism (called an orientation of G) and all $\alpha_g \in R$. Note that such a σ is uniquely determined by its kernel $\ker(\sigma) = N$.

The aim of this paper is to prove the following theorem in which we fully describe when $(RG)^-_{\varphi_\sigma}$ is commutative, and this in terms of presentations of the groups G and kernels N. Because of the results mentioned above, we will only deal with the case that $G \neq \ker(\sigma)$ and therefore $\mathrm{char}(R) \neq 2$. Moreover, if $\mathrm{char}(R) = 2$ then the antisymmetric elements are precisely the symmetric elements and in [3, 12] it has been classified when the symmetric elements in RG commute.

We will denote by $R_2 = \{r \in R \mid 2r = 0\}$.

Theorem 1.1 *Let R be a commutative ring. Let G be a nonabelian group with a nontrivial orientation homomorphism σ. Let $N = Ker(\sigma)$ and denote by E an elementary abelian 2-group. Then, $(RG)^-_{\varphi_\sigma}$ is commutative if and only if one of the following conditions holds*

1. $R_2 = \{0\}$, $G = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ ab = ba^3\rangle \times E$ *and* $N = \langle a^2, ab\rangle \times E$*;*

2. $\mathrm{char}(R) = 4$ *and* $G = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ ab = ba^{-1}\rangle \times E$, $N = \langle a^2, b\rangle \times E$ *or* $N = \langle a^2, ab\rangle \times E$*;*

3. $R_2 = \{0\}$ *and N is an elementary abelian 2-group;*

4. *G is a Hamiltonian 2-group and one of the following conditions is satisfied:*

   (i) N is abelian,

   (ii) N is a Hamiltonian 2-group and $\mathrm{char}(R) = 4$*;*

5. $G = \langle a, b \mid a^4 = b^4 = 1,\ ab = b^{-1}a\rangle \times E$ *and* N *is equal to either* $\langle a, b^2\rangle$ *or* $\langle ab, b^2\rangle \times E$,

6. $\mathrm{char}(R) = 4$, $G = \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = a^2,\ ab = ba,\ ac = ca^{-1},\ bc = cb^{-1}\rangle \times E$ *and* N *is equal to either* $\langle a, c\rangle \times \langle b^2\rangle \times E$ *or* $\langle a, bc\rangle \times \langle b^2\rangle \times E$;

7. $R_2 = \{0\}$, $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab\rangle \times E$ *and* N *is equal to either* $\langle a, b\rangle \times E$, $\langle a, c\rangle \times E$ *or* $\langle b, c\rangle \times E$;

8. $R_2 = \{0\}$, $G = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1,\ ab = ba,\ ac = ca,\ ad = dab,\ bc = cb,\ bd = db,\ cd = da^2c\rangle \times E$ *and* $N = \langle b\rangle \times \langle c, d\rangle \times E$;

9. $R_2 = \{0\}$, $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ ac = ca^{-1},\ bc = ca^2b^{-1}\rangle \times E$ *and* $N = \langle a, c\rangle \times \langle b^2\rangle \times E$;

10. $R_2 = \{0\}$, $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ ac = ca,\ bc = ca^2b\rangle \times E$ *and* $N = \langle b, c\rangle \times E$ *or* $N = \langle ab, c\rangle \times E$.

The outline of the paper is as follows. In Section 2 we give several examples, and in particular the sufficiency of the conditions in the theorem follow. In Section 3 we prove several technical lemmas. It follows that if $(RG)^-_{\varphi_\sigma}$ is commutative then the exponent of G divides 8. In Section 4 we deal with groups of exponent 8 (this corresponds with cases 1 and 2 of Theorem 1.1). In Section 5 we handle groups of exponent 4 and abelian kernel N (this corresponds with cases 3, 4 with abelian kernel and 5 of Theorem 1.1). Finally in Section 6 the remaining cases are dealt with, that is, G has exponent 4 and N is not abelian (this corresponds with cases 4 with N a Hamiltonian 2-group and cases 6 to 10 of Theorem 1.1)).

2 Sufficient Conditions

In this section we give several examples of finite groups G with a nontrivial orientation morphism $\sigma : G \to \{-1, 1\}$ so that $(RG)^-_{\varphi_\sigma}$ is commutative for any commutative ring R. These examples are needed to prove the sufficiency of the conditions in the main result.

Throughout R is a commutative ring of characteristic not 2 and G is a group with nontrivial orientation morphism σ. The classical involution on G is denoted by $*$. The order of $g \in G$ is denoted by $\circ(g)$ and the center of G is denoted by $\mathcal{Z}(G)$. For subsets X and Y of a ring T we denote by $[X, Y]$ the set of commutators $[x, y] = xy - yx$ with $x \in X$ and $y \in Y$, and the multiplicative commutator $ghg^{-1}h^{-1}$ of $g, h \in G$ is denoted by (g, h).

The kernel of σ will always be denoted by N and by assumption it always is a proper subgroup of G. So, N is a subgroup of index 2 in G. It is obvious that the involution φ_σ coincides on the subring RN with the ring involution $*$ and that the antisymmetric elements in G, under φ_σ, are the symmetric elements in $G \setminus N$ under $*$. Then as an R-module, $(RG)^-_{\varphi_\sigma}$ is generated by the set

$$\begin{aligned}\mathcal{S} \;=\; & \{g \in G \setminus N \mid g^2 = 1\} \cup \{g - g^{-1} \mid g \in N\} \cup \{g + g^{-1} \mid g \in (G \setminus N),\ g^2 \neq 1\} \\ & \cup \{rg \mid g \in N,\ g^2 = 1 \text{ and } r \in R_2\}.\end{aligned}$$

We begin with stating an obvious but useful remark.

Remark 2.1 *Let $G = H \times E$, a direct product of groups, with E an elementary abelian 2-group. Let σ and σ_1 be orientation homomorphisms of G and H with kernels N and N_1, respectively. If $N = N_1 \times E$ then $(RG)_{\varphi_\sigma}$ is commutative if and only if $(RH)^-_{\varphi_{\sigma_1}}$ is commutative.*

Our first example is that of Hamiltonian 2-groups.

Proposition 2.2 *If G is a Hamiltonian 2-group then*

1. *If N is abelian $(RG)^-_{\varphi_\sigma}$ is commutative.*

2. *If N is not abelian and* $\mathrm{char}(R) = 4$ *abelian then $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. 1. Assume N is abelian. Because of Remark 2.1, it is sufficient to deal with the case $G = Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$ and $N = \langle a \rangle$. Because elements of order 2 are central in G, we only need to check that $[a - a^{-1}, x + x^{-1}] = 0$ and $[y + y^{-1}, x + x^{-1}] = 0$ for all $x, y \notin N$. For the former we may assume that $x = b$. Since $ab = a^{-1}b^{-1}$, $ab^{-1} = a^{-1}b$, $ba = b^{-1}a^{-1}$ and $ba^{-1} = b^{-1}a$ we get that

$$\begin{aligned}
[a - a^{-1}, b + b^{-1}] &= ab + ab^{-1} - a^{-1}b - a^{-1}b^{-1} \\
&\quad -ba + ba^{-1} - b^{-1}a + b^{-1}a^{-1} = 0,
\end{aligned}$$

as desired.

For the remaining case it is sufficient to deal with $x = b$ and $y = ab$. Now

$$\begin{aligned}
[ab + b^{-1}a^{-1}, b + b^{-1}] &= ab^2 + a + b^{-1}a^{-1}b + b^{-1}a^{-1}b^{-1} \\
&\quad -bab - a^{-1} - b^{-1}ab - b^{-2}a^{-1} \\
&= a^{-1} + a + a + a^{-1} - a - a^{-1} - a^{-1} - a = 0,
\end{aligned}$$

again as desired.

2. Assume that $\mathrm{char}(R) = 4$ and N is not abelian, i.e. it is Hamiltonian 2-group. Then $G = N \times E$ with E a cyclic group of order 2. It is easily checked that the antisymmetric elements in RN commute. This also follows from Example 4.1 in [2] (one uses that $\mathrm{char}(R) = 4$). Since E is central in G it then also is easily checked that $(RG)^-_{\varphi_\sigma}$ is commutative. ∎

Next we deal with four groups of order 16. We will write $G_{[a,b]}$ to denote the group $[a, b]$ in The Small Group library in GAP [6].

Proposition 2.3 *Let $G = G_{[16,8]} = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ ab = ba^3\rangle = \langle a \rangle \cup \langle a \rangle b$ and R a commutative ring with $R_2 = \{0\}$. Then, $N = \langle a^2, ab\rangle = \langle a^2 \rangle \cup \langle a^2 \rangle ab$ is the only proper kernel for which $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. The only subgroups of index 2 in G are $\langle a \rangle$, $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$. In the first two cases we have that $ab, a^3b \in (RG)^-_{\varphi_\sigma}$, but these elements do not commute.

So the only possible kernel is $N = \langle a^2, ab\rangle = \{1, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}$. We need to show that then $(RG)^-_{\varphi_\sigma}$ is commutative. Since $R_2 = \{0\}$ and the only elements of order 2 are a^4, ab, a^3b, a^5b and a^7b, it is enough to show that $[A, A] = 0$ with $A = \{a^2 - a^6,\ a + a^{-1},\ a^3 + a^5,\ b + b^{-1},\ a^2b + a^6b\}$. Because $\langle a^2, b\rangle \cong Q_8$ it is sufficient to check that $[a + a^{-1}, b + b^{-1}] = 0$. As $\langle a, b\rangle = \langle a, a^2b\rangle$ and $\circ(a^2b) = 4$, it follows that $[a + a^{-1}, a^2b + (a^2b)^{-1}] = [a + a^{-1}, a^2b + a^6b] = 0$. Hence the result follows. Therefore $a \notin N$. ∎

Proposition 2.4 *Let $G = G_{[16,9]} = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ ab = ba^{-1}\rangle = \langle a \rangle \cup \langle a \rangle b$ and R a commutative ring with* $\mathrm{char}(R) = 4$*. Then $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$ are the only kernels N for which $(RG)^-_{\varphi_\sigma}$ is commutative. Therefore $a \notin N$.*

Proof. Let $N = \langle a^2, b\rangle = \langle a^2\rangle \cup \langle a^2\rangle b$. Since $\mathrm{char}(R) = 4$ and $N \cong Q_8$, one easily checks that $(RN)^-$ is commutative (or see [2]). Let $A = \{a + a^{-1}, a^3 + a^{-3}, ab + ab^{-1}, a^3b + a^{-1}b\}$ and $B = \{a^2 - a^{-2}, b - b^{-1}, a^2b - a^{-2}b\}$. Because a^4 is the only element of order 2 in G and a^4 is central, it is enough to show that $[A, A] = 0$ and $[A, B] = 0$. Clearly the elements that only depend on a commute. Also $[ab + ab^{-1}, a^3b + a^{-1}b] = [ab + ab^{-1}, a^{-1}b^{-1} + a^{-1}b] = (2 - 2)a^2 + (2 - 2)a^{-2} = 0$.

Now notice that if $g, h \in G$ are such that $o(g) = 8$, $h^2 = g^4$ and $gh = hg^{-1}$ then $[g + g^{-1}, h \pm h^{-1}] = [g + g^{-1}, h \pm hg^4] = 0$. Thus, since $G = \langle a, ab\rangle = \langle a, a^3b\rangle = \langle a^3, ab\rangle = \langle a^3, a^3b\rangle$, we get that $[A, A] = 0$. As $G = \langle a, b\rangle = \langle a, a^2b\rangle = \langle a^3, b\rangle = \langle a^3, a^2b\rangle$ we obtain that $0 = [a + a^{-1}, b - b^{-1}] = [a + a^{-1}, a^2b - a^{-2}b] = [a^3 + a^{-3}, b - b^{-1}] = [a^3 + a^{-3}, a^2b - a^{-2}b]$. Finally, if $g, h \in G$ are such that $o(g) = 4 = o(h)$ then $g^2 = h^2$. Therefore $h^{-1} = g^2h = hg^2$ and $[g + g^{-1}, h - h^{-1}] = 0$ and hence $[A, B] = 0$.

Replacing b by ab we also get the result for $N = \langle a^2, ab\rangle$. ∎

Proposition 2.5 *Let $G = G_{[16,13]} = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab\rangle = \langle a, b\rangle \cup \langle a, b\rangle c$ and R a commutative ring with $R_2 = \{0\}$. Then the only kernels $N = \ker(\sigma)$ for which $(RG)^-_{\varphi_\sigma}$ is commutative are $N = \langle a, b\rangle, \langle a, c\rangle$ and $\langle b, c\rangle$. (Note that these are all the nonabelian subgroups isomorphic to D_4.)*

Proof. Note that $\mathcal{Z}(G) = \langle abc\rangle = \{1, abc, (ab)^2, bac\}$ and G is of exponent 4.

First we show that $(RG)^-_{\varphi_\sigma}$ is commutative with $N = \langle a, b\rangle$. Since $R_2 = \{0\}$ and ab and ba are the only elements of order 4 in N, it is enough to show that $[ab - ba, A] = 0$ and $[A, A] = 0$, where $A = \{c, (ab)^2c, ac+ca, bc+cb\}$. Let $x = abc \in \mathcal{Z}(G)$. Then, $x^{-1} = bac$ and $ab-ba = (x-x^{-1})c$. Thus clearly $ab-ba$ commutes with c and $x^2c = (ab)^2c$. Moreover, since $ac+ca = b(bac+bca) = b(x^{-1}+x)$, $bc + cb = bc + (bc)^{-1} = bc + abac = a(x + x^{-1})$ and $(x - x^{-1})(x + x^{-1}) = x^2 - x^{-2} = 0$, we have that $[ab - ba, A] = 0$. As $(ab)^2$ is central, it is easy to see that c and $(ab)^2c$ commute with $ac + ca$ and $bc + cb$. Finally, $[ac + ca, bc + cb] = (x + x^{-1})^2[b, a] = (x + x^{-1})^2(x^{-1} - x)c = 0$ and $[A, A] = 0$. Therefore $(RG)^-_{\varphi_\sigma}$ indeed is commutative.

Analogously, due to the symmetry in the generators of G, we have that if N is equal to either $\langle a, c\rangle$ or $\langle b, c\rangle$ then $(RG)^-_{\varphi_\sigma}$ is commutative.

Notice that G has four other possible kernels: $N_1 = \langle a, bc\rangle$, $N_2 = \langle b, ac\rangle$, $N_3 = \langle c, ab\rangle$ and $N_4 = \langle ab, ac\rangle$. If $N = N_1$ then $b, c \in (RG)^-_{\varphi_\sigma}$ and they do not commute. If $N = N_2$ then $a, c \in (RG)^-_{\varphi_\sigma}$ and they do not commute. If $N = N_3$ then $b, a \in (RG)^-_{\varphi_\sigma}$ and they do not commute. Finally if $N = N_4$ then $b, a \in (RG)^-_{\varphi_\sigma}$ and they do not commute. ∎

Proposition 2.6 *Let $G = G_{[16,4]} = \langle a, b \mid a^4 = b^4 = 1,\ ab = b^{-1}a\rangle$. Then, $\langle a, b^2\rangle$ and $\langle ab, b^2\rangle$ are the only kernels $N = \ker(\sigma)$ for which $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. Notice that $\mathcal{Z}(G) = \langle a^2\rangle \times \langle b^2\rangle$ and that the only subgroups of index 2 in G are $\langle a, b^2\rangle$, $\langle ab, b^2\rangle$ and $\langle a^2, b\rangle$.

First assume $N = \langle a, b^2\rangle = \langle a\rangle \times \langle b^2\rangle$. Since N is abelian and elements of order 2 are central, to prove that $(RG)^-_{\varphi_\sigma}$ is commutative, it is enough to show that $[A, A] = 0$ and $[A, B] = 0$, where $A = \{b + b^{-1}, ab + a^{-1}b, ab^{-1} + a^{-1}b^{-1}, a^2b + a^2b^{-1}\}$ and $B = \{a - a^{-1}, ab^2 - a^{-1}b^2\}$. Now, since $b+b^{-1} = (1+b^2)b$ and $ab+a^{-1}b = (1+a^2)ab$ we have that $[b+b^{-1}, ab+a^{-1}b] = (1+a^2)(1+b^2)[b, ab] = (1 + a^2)(1 + b^2)a(1 - b^2) = a(1 + a^2)(1 - b^4) = 0$. Similarly, since $ab^{-1} + a^{-1}b^{-1} = (1 + a^2)ab^{-1}$, $a^2b + a^2b^{-1} = a^2(1 + b^2)b$, $[b, ab^{-1}] = -a(1 - b^2)$ and $[ab, ab^{-1}] = 0$ we have that $[A, A] = 0$. On the other hand, since $a - a^{-1} = a(1 - a^2)$, $ab^2 - a^{-1}b^2 = a(1 - a^2)b^2$ and $(1 - a^4) = 0$ we have that the elements in $[\{ab + a^{-1}b,\ ab^{-1} + a^{-1}b^{-1}\}, B] = 0$. Moreover, since $[a, b] = ab(1 - b^2)$ and

$(1-b^4)=0$ we have that $[\{b+b^{-1},\ a^2b+a^2b^{-1}\},B]=0$. Thus $[A,B]=0$ and therefore $(RG)^-_{\varphi_\sigma}$ indeed is commutative.

Replacing a by ab we also get that $(RG)^-_{\varphi_\sigma}$ is commutative for $N=\langle ab,b^2\rangle$ then $(RG)^-_{\varphi_\sigma}$ is commutative.

On the other hand, if $N=\langle b,a^2\rangle=\langle b\rangle\times\langle a^2\rangle$ then $(RG)^-_{\varphi_\sigma}$ is not commutative because

$$\begin{aligned}[a+a^{-1},b-b^{-1}] &= (1+a^2)(1-b^2)[a,b]=(1+a^2)(1-b^2)^2ab \\ &= (1+a^2)2(1-b^2)ab=2(1+a^2-b^2-a^2b^2)ab \\ &= 2(ab+a^{-1}b-ab^{-1}-a^{-1}b^{-1})\neq 0.\end{aligned}$$

∎

Thirdly we deal with four groups of order 32.

Proposition 2.7 *Let $G=G_{[32,35]}=\langle a,b,c\mid a^4=b^4=1,\ c^2=a^2,\ ab=ba,\ ac=ca^{-1},\ bc=cb^{-1}\rangle$ and R a commutative ring with $\mathrm{char}(R)=4$. Then the only kernels $N=\ker(\sigma)$ for which $(RG)^-_{\varphi_\sigma}$ is commutative are $\langle a,c\rangle\times\langle b^2\rangle$ and $\langle a,bc\rangle\times\langle b^2\rangle$.*

Proof. First, notice that $\mathcal{Z}(G)=\langle a^2\rangle\times\langle b^2\rangle$. Let $N=\langle a,c\rangle\times\langle b^2\rangle$. Then, since $\mathrm{char}(R)=4$ and $N\cong Q_8\times C_2$, we have that $(RN)^-_{\varphi_\sigma}$ is commutative (see [2]). Let $A_0=\{b+b^3\}$, $A_1=\{a^3b+ab^3, bc+a^2bc, abc-a^3bc\}$, $B_0=\{a-a^3, c-a^2c, ac+a^3c\}$, $A=A_0\cup A_0a^2\cup A_1\cup A_1b^2$ and $B=B_0\cup B_0b^2$. To prove that $(RG)^-_{\varphi_\sigma}$ is commutative, it is enough to show $[A,A]=[B,B]=[A,B]=0$. Since a^2 and b^2 are central, it is enough to show $[A_0\cup A_1,A_0\cup A_1]=[B_0,B_0]=[A_0\cup A_1,B_0]=0$. Again, as a^2 and b^2 are central, we can write $b+b^3=(1+b^2)b$, $a^3b+ab^3=(a^2+b^2)ab$, $bc+a^2bc=(1+a^2)bc$, and $abc+a^3bc=(1+a^2)abc$. Thus, since $ab=ba$, it follows that $[b+b^3,a^3b+ab^3]=0$. Also, $[b+b^3,bc+a^2bc]=(1+a^2)(1+b^2)[b,bc]=(1+a^2)(1+b^2)(b^2-1)c=0$. Similarly, as $[b,abc]=(b^2-1)ac$, $[ab,bc]=(b^2-a^2)ac$ and $[ab,abc]=a^2b^2c-abcab=(a^2b^2-1)c$, we get that $[b+b^3,abc+a^3bc]=0$ and $[a^3b+ab^3,\{bc+a^2bc,\ abc+a^3bc\}]=0$. Moreover, as $[bc,abc]=(1-a^2)a$ and $(1+a^2)(1-a^2)=0$, we obtain that $[bc+a^2bc,abc+a^3bc]=0$. Therefore $[A_0\cup A_1,A_0\cup A_1]=0$.

Because $\mathrm{char}(R)=4$, we also obtain that $(1-a^2)^3=0$. It then easily follows that the elements of $[B_0,B_0]=0$.

On the other hand, since $a-a^3=(1-a^2)a$, we have that $[a-a^3,A_0\cup A_1]=0$. Also, as $c-a^2c=(1-a^2)c$, $[c,b]=(b^2-1)bc$, $[c,ab]=(a^2b^2-1)abc$ and $(1+b^2)(1-b^2)=0=(1+a^2b^2)(1-a^2b^2)$, it follows that $[c-a^2c,A_0\cup A_1]=0$. Similarly, as $ac-a^3c=(1-a^2)ac$, $[ac,b]=(b^2-1)abc$ and $[ac,ab]=(b^2-a^2)bc$, we have that $[ac-a^3c,A_0\cup A_1]=0$. Therefore the elements of $[A_0\cup A_1,B_0]=0$. Hence $(RG)^-_{\varphi_\sigma}$ is commutative.

Similarly, replacing c by bc, we get that if $\mathrm{char}(R)=4$ and $N=\langle a,bc\rangle\times\langle b^2\rangle$ then $(RG)^-_{\varphi_\sigma}$ is commutative.

Notice that G has five other possible kernels: $N_1=\langle a,b\rangle$, $N_2=\langle a^2,b,c\rangle$, $N_3=\langle a^2,b,ac\rangle$, $N_4=\langle a^2,a^3b,c\rangle$ and $N_5=\langle a^2,a^3b,ac\rangle$. But, if $N=N_1$, N_2 or N_3 then $b-b^3\in(RG)^-_{\varphi_\sigma}$ and either $c-a^2c\in(RG)^-_{\varphi_\sigma}$ or $c+a^2c\in(RG)^-_{\varphi_\sigma}$, and $[b-b^3,c\mp a^2c]=-2(1\mp a^2)(1-b^2)bc\neq 0$. On the other hand, if $N=N_4$ or N_5 then $ab-a^3b^3\in(RG)^-_{\varphi_\sigma}$ and either $c-a^2c\in(RG)^-_{\varphi_\sigma}$ or $c+a^2c\in(RG)^-_{\varphi_\sigma}$, and $[ab-a^3b^3,c\mp a^2c]=2(1\mp a^2)(1-a^2b^2)abc\neq 0$. ∎

Proposition 2.8 *Let $G=G_{[32,30]}=\langle a,b,c,d\mid a^4=b^2=c^2=d^2=1,\ ab=ba,\ ac=ca,\ ad=dab,\ bc=cb,\ bd=db,\ cd=da^2c\rangle$ and R a commutative ring with $R_2=\{0\}$. Then, $N=\langle b\rangle\times\langle c,d\rangle$ is the only kernel N for which $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. Notice $\mathcal{Z}(G) = \langle a^2\rangle \times \langle b\rangle$. Assume $N = \langle b\rangle \times \langle c, d\rangle$. Let $A_0 = \{a + a^3, ac + a^3c, ad + a^3bd\}$, $A_1 = \{acd+abcd\}$, $B_0 = \{cd-a^2cd\}$, $A = A_0 \cup A_0b \cup A_1 \cup A_1a^2$ and $B = B_0 \cup B_0b$. Since $R_2 = \{0\}$, to prove that $(RG)^-_{\varphi_\sigma}$ is commutative, it is enough to show that $[A, A] = [B, B] = [A, B] = 0$. Clearly, as b is central, the elements of $[B, B] = 0$. Moreover, since a^2 is central, we can write $a + a^3 = (1 + a^2)a$, $ac + a^3c = (1 + a^2)ac$ and $acd + abcd = (1 + b)acd$. Thus, since $ac = ca$, we get that $[a + a^3, ac + a^3c] = 0$. Also, as $[a, acd] = (1 - b)a^2cd$ and $(1 + b)(1 - b) = 0$, we have that $[a+a^3, acd+acdb] = 0$. Now, $[a+a^3, ad+a^3bd] = (1+a^2)(1+b)[a, ad] = (1+a^2)(1+b)(1-b)a^2d = 0$. Similarly, as $(1 + a^2)(1 + a^2b) = (1 + b)(1 + a^2b)$, $[ac, ad] = (a^2 - b)cd$, $[ac, acd] = (a^2 - b)d$ and $[ad, acd] = (1 - a^2)bc$, we have that $[a + a^3, ad + a^3bd] = 0$, $[ac + a^3c, \{ad + a^3bd,\ acd + abcd] = 0$ and $[ad + a^3bd, acd + abcd] = 0$. Thus, $[A_0 \cup A_1, A_0 \cup A_1] = 0$ and thus $[A, A] = 0$. Since $cd-a^2cd = (1-a^2)cd$, $0 = (1+a^2)(1-a^2) = (1+a^2b)(1-a^2b) = (1-b)(1+b)$, $[ad, cd] = a^3c(1-a^2b)$ and $[acd, cd] = a^3(1 - b)$, we have that $[A_0 \cup A_1, B_0] = 0$. Therefore the elements of $[A, B] = 0$. Hence $(RG)^-_{\varphi_\sigma}$ is commutative.

Now, notice that G has six other possible kernels: $N_1 = \langle a, b, c\rangle$, $N_2 = \langle a^2, b, c, a^3bd\rangle$, $N_3 = \langle a, b, d\rangle$, $N_4 = \langle a^2, b, a^3c, d\rangle$, $N_5 = \langle a, b, a^2cd\rangle$ and $N_6 = \langle a^2, b, a^3c, a^3bd\rangle$. If $N = N_1$ or N_2 then $d \in (RG)^-_{\varphi_\sigma}$ and either $a - a^3 \in (RG)^-_{\varphi_\sigma}$ or $a + a^3 \in (RG)^-_{\varphi_\sigma}$, but $[a \mp a^3, d] = (1 \mp a^2)(b - 1)da \neq 0$. If $N = N_3$ or N_4 then $c \in (RG)^-_{\varphi_\sigma}$ and either $ad - a^3bd \in (RG)^-_{\varphi_\sigma}$ or $ad + a^3bd \in (RG)^-_{\varphi_\sigma}$, but $[c, ad \mp a^3bd] = (1 \mp a^2b)(1 - a^2)acd \neq 0$. Finally, if $N = N_5$ or N_6 then $c, d \in (RG)^-_{\varphi_\sigma}$ and they do not commute. ∎

Proposition 2.9 *Let $G = G_{[32,31]} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ ac = ca^{-1},\ bc = ca^2b^{-1}\rangle$ and R a commutative ring with $R_2 = \{0\}$. Then, $\langle a, c\rangle \times \langle b^2\rangle$ is the only kernel for which $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. Notice that $\mathcal{Z}(G) = \langle a^2\rangle \times \langle b^2\rangle$. Assume $N = \langle a, c\rangle \times \langle b^2\rangle$. Let $A_0 = \{b + b^3\}$, $A_1 = \{ab + a^3b^3,\ bc + a^2bc,\ abc + a^3bc\}$, $B_0 = \{a - a^3\}$, $A = A_0 \cup A_0a^2 \cup A_1 \cup A_1b^2$ and $B = B_0 \cup B_0b^2$. Since $R_2 = \{0\}$, to prove that $(RG)^-_{\varphi_\sigma}$ is commutative it is enough to show that $[A, A] = [B, B] = [A, B] = 0$. Clearly, as b^2 is central, it follows that $[B, B] = 0$. Since a^2 is central, we can write $A_1 = \{ab(1 + a^2b^2),\ (1 + a^2)bc,\ (1 + a^2)abc\}$. Thus, because $ab = ba$, it follows that $[b + b^3, ab + a^3b^3] = 0$. Also, as $[bc, abc] = a(1 - a^2)$ and $(1 + a^2)(1 - a^2) = 0$, we have that $[bc+a^2bc, abc+a^3bc] = 0$. On the other hand, as $(1+a^2)(1+b^2) = (1+a^2)(1+a^2b^2)$, $[b, bc] = (b^2-a^2)c$, $[b, abc] = (b^2 - a^2)ac$, $[bc, ab] = 0$ and $[ab, abc] = (b^2 - 1)a^2c$, we have that $[A_0 \cup A_1, A_0 \cap A_1] = 0$ and thus $[A, A] = 0$.

As $a - a^3 = (1 - a^2)a$, $ab = ba$ and $(1 + a^2)(1 - a^2) = 0$, we have $[A_0 \cup A_a, B_0] = 0$. Therefore $[A, B] = 0$ and hence $(RG)^-_{\varphi_\sigma}$ is commutative.

Notice that G has six other possible kernels: $N_1 = \langle a^2, b, c\rangle$, $N_2 = \langle a^2, a^2b, c\rangle$, $N_3 = \langle a^2, b, ac\rangle$, $N_4 = \langle a^2, a^3b, ac\rangle$, $N_5 = \langle a, b^2, b^2c\rangle$ and $N_6 = \langle a, b\rangle$. If $N = N_1$ or N_2 then $ac \in (RG)^-_{\varphi_\sigma}$ and either $b - b^3 \in (RG)^-_{\varphi_\sigma}$ or $b + b^3 \in (RG)^-_{\varphi_\sigma}$, but $[ac, b \mp b^3] = (1 \mp b^2)(a^2b^2 - 1)abc \neq 0$. If $N = N_3$ or N_4 then $c \in (RG)^-_{\varphi_\sigma}$ and and either $b - b^3 \in (RG)^-_{\varphi_\sigma}$ or $b + b^3 \in (RG)^-_{\varphi_\sigma}$, but $[c, b \mp b^3] = (1 \mp b^2)(a^2b^2 - 1)bc \neq 0$. If $N = N_5$ or N_6 then $c, ac \in (RG)^-_{\varphi_\sigma}$ and they do not commute. ∎

Proposition 2.10 *Let $G = G_{[32,24]} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ ac = ca,\ bc = ca^2b\rangle$ and R a commutative ring with $R_2 = \{0\}$. Then, the only kernels for which $(RG)^-_{\varphi_\sigma}$ is commutative are $\langle b, c\rangle$ and $\langle ab, c\rangle$.*

Proof. Notice that $\mathcal{Z}(G) = \langle a\rangle \times \langle b^2\rangle$. Assume $N = \langle b, c\rangle$. Then, since N contains an elementary abelian 2-subgroup of index 2, it follows that $(RN)^-$ is commutative (see [2]). Now, let $A_0 = \{ac + a^3c, ab + a^3b^3\}$, $A_1 = \{abc + ab^3c\}$, $B_0 = \{b - b^3\}$, $B_1 = \{bc - a^2b^3c\}$, $A = A_0 \cup A_0b^2 \cup A_1 \cup A_1a^2$ and $B = B_0 \cup B_0a^2 \cup B_1 \cup B_1b^2$. Since $R_2 = \{0\}$, to prove that $(RG)^-_{\varphi_\sigma}$ is commutative, it is enough to show $[A, A] = [A, B] = [B_0, B_1] = 0$. The last equality follows from $[(1 - b^2)b, (1 - a^2b^2)bc] = (1 - b^2)(1 - a^2b^2)[b, bc] = (1 - b^2)(1 - a^2b^2)(a^2 - 1)c = 0$. Since a^2 and b^2 are central, we can write $ab + a^3b^3 = a(1 + a^2b^2)b$, $ac + a^3c = a(1 + a^2)c$, $abc + ab^3c = a(1 + b^2)bc$, $b - b^3 = (1 - b^2)b$ and $bc - a^2b^3c = (1 - a^2b^2)bc$. Thus $[ab + a^3b^3, b - b^3] = 0$ and, as $(1 + a^2b^2)(1 - a^2b^2) = 0$, $[ab + a^3b^3, bc - a^2b^3c] = 0$. Also, since $[b, bc] = (1 - a^2)b^2c$ and $(1 + a^2b^2)(1 + b^2)(1 - a^2) = 0$, we have that $[ab + a^3b^3 = (1 + a^2b^2)ab, abc + ab^3c = (1 + b^2)abc] = 0$. On the other hand, since $(1 + a^2b^2)(1 - a^2b^2) = 0$, $[c, b] = (a^2 - 1)bc$ and $[c, bc] = (a^2 - 1)b$, we have that $[ac + a^3c, \{ab + a^3b^3,\ abc + ab^3c,\ b - b^3,\ bc - a^2b^3c\}] = 0$. Finally, notice that $[abc + ab^3c, bc - a^2b^3c] = 0$ and, as $(1+b^2)(1-b^2) = 0$, $[abc+ab^3c, b-b^3] = 0$. Therefore, $[A_0 \cup A_1, A_0 \cup A_1] = 0$ and $[A_0 \cup A_1, B_0 \cup B_1] = 0$. Hence $(RG)^-_{\varphi_\sigma}$ is commutative.

Notice that $G = \langle a, ab, c\rangle$, $o(ab) = 4$, $a(ab) = (ab)a$ and $(ab)c = ca^2(ab)^{-1}$. Hence, replacing b by ab, we get that also $N = \langle ab, c\rangle$ is a kernel.

Finally, G has five other possible kernels: $\langle a, b^2, c\rangle$, $\langle a, b\rangle$, $\langle a^2, b, a^3c\rangle$, $\langle a, b^2, a^2b^3c\rangle$ and $\langle a^2, a^3b, a^3c\rangle$. If $N = \langle a, b^2, c\rangle$ then $ac - a^3c$, $b + b^3 \in (RG)^-_{\varphi_\sigma}$ and they do not commute. Otherwise, $c \in (RG)^-_{\varphi_\sigma}$ and either $b - b^3 \in (RG)^-_{\varphi_\sigma}$ or $b + b^3 \in (RG)^-_{\varphi_\sigma}$, but $[c, b \pm b^3] = (a^2 - 1)(1 \pm b^2)bc \neq 0$. ∎

We finish this section with one more elementary remark.

Remark 2.11 *Let G be a group and let A be a subgroup of index 2 in G. Assume that $A = C \times E$, a direct product of groups, with E an elementary abelian 2-group. If E is central in G then G is the central product of the subgroups E and $\langle C, g\rangle$, with $g \in G \setminus A$.*

3 Necessary conditions

We begin with a series of technical lemmas that yield necessary conditions for $(RG)^-_{\varphi_\sigma}$ to be commutative.

Lemma 3.1 *Assume that $(RG)^-_{\varphi_\sigma}$ is commutative.*

1. *If $R_2 = \{0\}$ and $g \in G \setminus N$ with $g^2 = 1$ then $(g, h) = 1$ for all $h \in G \setminus N$ with $h^2 = 1$ and for all $h \in N$ with $h^2 \neq 1$.*

2. *If $R_2 \neq \{0\}$ and $g \in G$ with $g^2 = 1$ then $g \in \mathcal{Z}(G)$.*

Proof. 1. Assume that $R_2 = \{0\}$ and $g \in G \setminus N$ with $g^2 = 1$. If $h \in G \setminus N$ with $h^2 = 1$, then g and h are two antisymmetric elements and hence $[g, h] = 0$ as desired. Assume now that $h \in N$ with $h^2 \neq 1$. Then $0 = [g, h - h^{-1}] = gh - gh^{-1} - hg + h^{-1}g$ and therefore we have that gh is equal to either gh^{-1} or hg. The former is excluded since by assumption $h^2 \neq 1$. Thus $gh = hg$, as desired.

2. Assume that $g \in G$ with $g^2 = 1$ and $0 \neq r \in R_2$. Suppose first that $g \in N$. Let $h \in G$. We need to show that $(g, h) = 1$. If $h \in G \setminus N$ with $h^2 = 1$ then $0 = [rg, h] = rgh - rhg$ and therefore $gh = hg$. If $h \in G \setminus N$ with $h^2 \neq 1$ then $0 = [rg, h + h^{-1}] = r(gh + gh^{-1} + hg + h^{-1}g)$. Since $h^2 \neq 1$, we clearly have that $gh \neq gh^{-1}$. Hence either $gh = hg$ (as desired) or $gh = h^{-1}g$. The latter implies that $(gh)^2 = 1$ with $gh \in G \setminus N$. So, by the previous case, $1 = (g, gh) = (g, h)$ as desired. Finally if $h \in N$ then choose $x \in G \setminus N$. By the previous, $(g, x) = 1 = (g, hx)$. Hence $1 = (g, h)$ as desired.

Second, assume that $g \in G \backslash N$. Let $h \in G \backslash N$. If $h^2 = 1$ then, by the above, $0 = [g, h]$, as desired. If $h^2 \neq 1$ then $0 = [g, h+h^{-1}] = gh+gh^{-1}-hg-h^{-1}g$. So either $gh = hg$ (as desired) or $gh = h^{-1}g$. The latter implies that$(gh)^2 = 1$ and $gh \in N$. So, by the above, $1 = (gh, h)$ and thus $1 = (g, h)$. We thus have shown that g commutes with all elements $h \in G \setminus N$. Assume now that $h \in N$. If $h^2 = 1$ then by the above $1 = (g, h)$, as desired. If $h^2 \neq 1$ then $0 = [g, h + h^{-1}] = gh + gh^{-1} - hg - h^{-1}g$. It follows that either $gh = hg$ (as desired) or $gh = h^{-1}g$ and thus $(gh)^2 = 1$ with $gh \in G \setminus N$ and by the above $1 = (g, gh) = (g, h)$ which finishes the proof of the lemma. ∎

Lemma 3.2 *Let g and h be elements of G with $g^2 \neq 1$ and $h^2 \neq 1$. If $[g - \varphi_\sigma(g), h - \varphi_\sigma(h)] = 0$ then the following properties hold.*

(*i*) *If $g, h \in N$ then one of the following conditions holds.*

- (*a*) $gh = hg$.
- (*b*) $R_2 = \{0\}$ *and* $(g^\alpha h^\beta)^2 = 1$, *for all* $\alpha, \beta \in \{-1, 1\}$.
- (*c*) $\mathrm{char}(R) = 4$ *and* $\langle g, h \rangle \cong Q_8$.

(*ii*) *If $g \in N$ and $h \notin N$ then one of the following conditions holds.*

- (*a*) $ghg^{-1} \in \{h, h^{-1}\}$.
- (*b*) $\circ(g) = 4 = \circ(h)$ *and* $g^2 = h^2$.

(*iii*) *If $g, h \notin N$ then one of the following conditions holds:*

- (*a*) $gh \in \{hg, h^{-1}g, hg^{-1}\}$.
- (*b*) $R_2 = \{0\}$ *and* $(g^\alpha h^\beta)^2 = 1$, *for all* $\alpha, \beta \in \{-1, 1\}$.

Proof. (*i*) By [2, Lemma 2.1] we have that either $gh = hg$; or $(g^\alpha h^\beta)^2 = 1$, for all $\alpha, \beta \in \{-1, 1\}$; or $\mathrm{char}(R) = 4$ and $\langle g, h \rangle \cong Q_8$. Notice that if $(g^\alpha h^\beta)^2 = 1$, for all $\alpha, \beta \in \{-1, 1\}$ and $R_2 \neq \{0\}$, then by Lemma 3.1 it follows that $1 = (gh, h) = (g, h)$ so we are in case (a).

(*ii*) Suppose $g \in N$ and $h \in G \backslash N$. Then, $0 = [g - g^{-1}, h + h^{-1}] = gh + gh^{-1} - g^{-1}h - g^{-1}h^{-1} - hg + hg^{-1} - h^{-1}g + h^{-1}g^{-1}$. As $g^2 \neq 1$, $h^2 \neq 1$ and $\mathrm{char}(R) \neq 2$, it follows that gh equals either hg, $h^{-1}g$, or $g^{-1}h^{-1}$.

Assume that $gh = g^{-1}h^{-1}$, that is, $g^2 = h^{-2}$. Then $0 = gh^{-1} - g^{-1}h + hg^{-1} - h^{-1}g$ and so gh^{-1} is equal to either $g^{-1}h$ or $h^{-1}g$. Therefore, $g^2 = h^2$ or $gh = hg$. But, if $g^2 = h^2$ then we obtain that $\circ(g) = 4 = \circ(h)$. Hence (ii) follows.

(*iii*) Suppose $g, h \notin N$. Then,

$$0 = [g + g^{-1}, h + h^{-1}] = gh + gh^{-1} + g^{-1}h + g^{-1}h^{-1} - hg - hg^{-1} - h^{-1}g - h^{-1}g^{-1}. \quad (1)$$

As $g^2 \neq 1$, $h^2 \neq 1$ and $\mathrm{char}(R) \neq 2$, it follows that gh equals either hg, $h^{-1}g$, hg^{-1}, or $h^{-1}g^{-1}$.

Assume that $gh = h^{-1}g^{-1}$, that is, $(gh)^2 = 1$, or equivalently $(g^{-1}h^{-1})^2 = 1$. If $R_2 \neq \{0\}$ then, by Lemma 3.1, it follows that $(g, h) = 1$. So assume that $R_2 = \{0\}$. By (1) we know that $0 = gh^{-1} + g^{-1}h - hg^{-1} - h^{-1}g$. Thus, either $gh^{-1} = hg^{-1}$ or $gh^{-1} = h^{-1}g$. Therefore, $(gh^{-1})^2 = 1$ and $(g^{-1}h)^2 = 1$, or $gh = hg$. This finishes the proof of the lemma. ∎

If $(RG)^-_{\varphi_\sigma}$ is commutative then the following remark can be applied to elements of order 2 that do not belong to N.

Lemma 3.3 *Assume that* $(RG)^-_{\varphi_\sigma}$ *is commutative with* $\mathrm{char}(R) \neq 2$*. Let* g *and* h *be noncommuting elements of* G *such that* $g^2 \neq 1$ *and* $h^2 \neq 1$*. The following properties hold.*

(i) *If* $g, h \in N$ *then one of the following conditions holds.*

  (a) $R_2 = \{0\}$ *and* $\langle g, h\rangle = \langle g, h \mid g^4 = h^4 = (gh)^2 = (gh^{-1})^2 = 1\rangle = G_{[16,3]}$.

  (b) $\mathrm{char}(R) = 4$ *and* $\langle g, h\rangle \cong Q_8$.

(ii) *If* $g \in N$ *and* $h \notin N$ *then one of the following conditions holds.*

  (a) $\langle g, h\rangle \cong G_{[16,4]}$.

  (b) $\langle g, h\rangle \cong Q_8$;

  (c) $\mathrm{char}(R) = 4$ *and* $\langle g, h\rangle \cong G_{[16,9]}$.

(iii) *If* $g, h \notin N$ *then one of the following conditions holds.*

  (a) $\langle g, h\rangle$ *is isomorphic to either* Q_8*, or* $G_{[16,4]}$.

  (b) $R_2 = \{0\}$ *and* $\langle g, h\rangle$ *is isomorphic to either* $G_{[16,3]}$*, or* $G_{[16,8]}$.

  (c) $\mathrm{char}(R) = 4$ *and* $\langle g, h\rangle \cong G_{[16,9]}$.

Proof. Assume $g, h \in G$ are noncommuting and $g^2 \neq 1$, $h^2 \neq 1$.

(i) Suppose $g, h \in N$. Because of Lemma 3.2 (i) we may assume that $R_2 = \{0\}$ and $(gh)^2 = (gh^{-1})^2 = 1$. Hence, it remains to show that $\circ(g) = 4$ and $\circ(h) = 4$. We prove the former, the latter is similar.

Let $x \in G \setminus N$. We may assume that $\circ(x) \neq 2$. Indeed, for assume $\circ(x) = 2$ then, by Lemma 3.1 (1), $gx = xg$. Thus $(gx)^2 = g^2 \neq 1$ and we may replace x by $gx \in G \setminus N$.

First we deal with the case that $gx \neq xg$. Applying Lemma 3.2 (ii) to the elements g and x, we get that either $gx = x^{-1}g$, or $\circ(g) = 4$ and $g^2 = x^2$. If $gx = x^{-1}g$ then, applying Lemma 3.2 (ii) to the elements g and gx, it follows that either $g^2x = (gx)^{-1}g = x^{-1}$ and hence $g^2 = x^{-2}$, or $\circ(g) = 4 = \circ(gx)$ and $g^2 = (gx)^2$. The former implies that $x^{-2} = gx^2g^{-1} = g^{-2} = x^2$ and thus $g^4 = x^{-4} = 1$, i.e. $\circ(g) = 4$. So we have shown that if $gx \neq xg$ then $\circ(g) = 4$.

Second we deal with the case that $gx = xg$. Because, by assumption $hg \neq gh$ and thus $g(hx) \neq (hx)g$, Lemma 3.1 (1) yields that $(hx)^2 \neq 1$. Thus, Lemma 3.2 (ii) applied to g and hx, gives that either $ghx = hxg = hgx$, $ghx = (hx)^{-1}g$, or $\circ(g) = 4$. The former is excluded because $gh \neq hg$. Since, $(gh)^2 = (gh^{-1})^2 = 1$ we also know that $ghx = h^{-1}g^{-1}x = h^{-1}xg^{-1}$ and $x^{-1}h^{-1}g = x^{-1}g^{-1}h = g^{-1}x^{-1}h$, the second option thus implies that $ghx = x^{-1}h^{-1}g = h^{-1}xg^{-1}$ and also $ghx = x^{-1}h^{-1}g = g^{-1}x^{-1}h$. Hence $g^{-2} = x^{-1}hx^{-1}h^{-1}$ and $g^2 = x^{-1}hx^{-1}h^{-1}$. So $g^4 = 1$. Therefore, we have shown that $gx = xg$ implies that $\circ(g) = 4$. This finishes the proof of (i).

(ii) Suppose $g \in N$ and $h \notin N$. Because of Lemma 3.2 (ii), we know that either $gh = h^{-1}g$, or $\circ(g) = 4 = \circ(h)$ and $g^2 = h^2$.

First, suppose that $gh = h^{-1}g$ and so g^2 is a central element in the group $\langle g, h\rangle$. Since $(gh)^2 = g^2 \neq 1$ we can apply Lemma 3.2 (ii) to the elements g and gh, and we obtain that either $g^2h = (gh)^{-1}g = h^{-1}$, or $\circ(g) = 4$. Thus, either $g^2 = h^{-2}$ or $\circ(g) = 4$. On the other hand, applying Lemma 3.2 (i) to g and $h^{-1}gh$, we have that either $gh^{-1}gh = h^{-1}ghg$; or $R_2 = \{0\}$ and $(g^{-1}h^{-1}gh)^2 = 1$, or $\mathrm{char}(R) = 4$ and $\langle g, h^{-1}gh\rangle \cong Q_8$. Consequently, since $gh = h^{-1}g$, we get that either $g^2h^2 = h^{-2}g^2 = g^2h^{-2}$; or $R_2 = \{0\}$ and $h^4 = 1$; or $\mathrm{char}(R) = 4$ and $g^2 = (gh^{-1}gh)^2 = (g^2h^2)^2 = g^4h^4$. Therefore, either $h^4 = 1$, or $\mathrm{char}(R) = 4$ and $g^2 = h^{-4}$. Hence $\langle g, h\rangle$ is isomorphic to either Q_8, $G_{[16,4]}$, or $\mathrm{char}(R) = 4$ and $\langle g, h\rangle \cong G_{[16,9]}$.

Second, suppose that $\circ(g) = 4 = \circ(h)$ and $g^2 = h^2$. Lemma 3.2 (i) applied to g and hgh yields that either $(gh)^2 = (hg)^2$; or $R_2 = \{0\}$ and $(gh)^4 = 1$; or $\mathrm{char}(R) = 4$ and $\langle g, hgh\rangle \cong Q_8$. Thus, either $(gh)^4 = 1$ or $\mathrm{char}(R) = 4$ and $g^2 = (gh)^4$. Therefore, either $\langle g, h\rangle \cong Q_8$, $\langle g, h\rangle = \langle g, h \mid g^4 = (gh)^4 = 1,\ g^2 = h^2\rangle \cong G_{[16,4]}$, or $\mathrm{char}(R) = 4$ and $\langle g, h\rangle = \langle g, h \mid g^4 = 1,\ (gh)^4 = g^2 = h^2\rangle \cong G_{[16,9]}$.

(iii) Suppose $g, h \notin N$. By Lemma 3.2 (iii) we have that either $gh = h^{-1}g$, $gh = hg^{-1}$, or $(g^\alpha h^\beta)^2 = 1$ for all $\alpha, \beta \in \{-1, 1\}$ and $R_2 = \{0\}$.

First, suppose that $gh = h^{-1}g$ and so g^2 is central in $\langle g, h\rangle$. Then, applying Lemma 3.2 (ii) to gh and g, we obtain that either $ghg = g^{-1}gh = h$ and hence $g^2 = h^2$, or $\circ(g) = 4$. On the other hand, applying Lemma 3.2 (i) to gh and hg, we get that either $gh^2g = hg^2h$; or $R_2 = \{0\}$ and $(gh(hg)^{-1})^2 = 1$; or $\mathrm{char}(R) = 4$ and $\langle gh, hg\rangle \cong Q_8$. Thus, we have that either $g^2h^{-2} = g^2h^2$; or $R_2 = \{0\}$ and $h^4 = 1$; or $\mathrm{char}(R) = 4$, $1 = (gh)^4 = g^4$ and $(gh)^2 = (hg^2h)^2 = (g^2h^2)^2 = g^4h^4$. Therefore, either $h^4 = 1$, or $\mathrm{char}(R) = 4$, $g^2 = h^4$ and $g^4 = 1$. Hence, either $\langle g, h\rangle \cong Q_8$, $\langle g, h\rangle \cong G_{[16,4]}$, or $\mathrm{char}(R) = 4$ and $\langle g, h\rangle \cong G_{[16,9]}$.

Second, suppose $gh = hg^{-1}$. Then the result follows at once from the previous case by replacing g by gh and h by g^{-1}.

Third, suppose that $R_2 = \{0\}$ and $(g^\alpha h^\beta)^2 = 1$, for all $\alpha, \beta \in \{-1, 1\}$. In particular, $hg^{-1}h = g$ and $ghg = h^{-1}$. Lemma 3.2 (iii) applied to ghg^{-1} and h yields that $((ghg^{-1})^\alpha h^\beta)^2 = 1$ for all $\alpha, \beta \in \{-1, 1\}$ (and, in particular, $(ghg^{-1}h)^2 = (g^2)^2 = g^4 = 1$) or $ghg^{-1}h = g^2$ is equal to either $hghg^{-1} = g^{-2}$, $h^{-1}ghg^{-1} = h^{-2}g^{-2}$ or $hgh^{-1}g^{-1} = h^2g^{-2}$. Thus, either $\circ(g) = 4$ or $g^4 = h^{\pm 2}$. Similarly, applying Lemma 3.2 (iii) to g and hgh^{-1}, it follows that either $\circ(h) = 4$ or $h^4 = g^{\pm 2}$.

If $\circ(g) = 4$ and $\circ(h) = 4$ then $\langle g, h\rangle \cong G_{[16,3]}$. On the other hand, if $\circ(g) = 4$ and $h^4 = g^2$, or $\circ(h) = 4$ and $g^4 = h^2$, then $\langle g, h\rangle \cong G_{[16,8]}$. Now, assume that $\circ(g) \neq 4$ and $\circ(h) \neq 4$. Then $g^4 = h^{\pm 2}$ and $h^4 = g^{\pm 2}$ and hence $g^2 = g^4g^{-2} = h^{\pm 2}h^{\mp 4} = h^{\mp 2}$. So $g^2h^{\pm 2} = 1$. On the other hand, since $(g^\alpha h^\beta)^2 = 1$, for all $\alpha, \beta \in \{-1, 1\}$, we have that $1 = g^2h^{\pm 2} = gh^{\mp 1}g^{-1}h^{\pm 1}$. Therefore $gh^{\mp 1} = h^{\mp 1}g$, and thus $gh = hg$, a contradiction. ∎

Lemma 3.4 *Assume that $(RG)^-_{\varphi_\sigma}$ is commutative. Let g and h be elements of G with $g^2 \neq 1$ and $h^2 = 1$. If $R_2 = \{0\}$ then the following properties hold.*

- (i) *If $g, h \in N$ then one of the following conditions holds.*
  - *(a) $\langle g, h\rangle$ is abelian.*
  - *(b) $(gh)^2 \neq 1$ and $\langle g, h\rangle \cong G_{[16,3]}$.*
  - *(c) $(gh)^2 = 1$ and $\langle g, h\rangle = \langle g, h \mid g^4 = h^2 = (gh)^2 = 1\rangle = D_4$.*
- (ii) *If $g \in N$ and $h \notin N$ then $\langle g, h\rangle$ is abelian.*
- (iii) *If $h \in N$ and $g \notin N$ then one of the following conditions holds.*
  - *(a) $\langle g, h\rangle$ is abelian.*
  - *(b) $(gh)^2 \neq 1$ and $\langle g, h\rangle$ is isomorphic to either $G_{[16,3]}$ or $G_{[16,8]}$.*
  - *(c) $(gh)^2 = 1$and $\langle g, h\rangle \cong D_4$.*
- (iv) *If $g, h \notin N$ then $\langle g, h\rangle$either is abelian or isomorphic to D_4.*

If $R_2 \neq \{0\}$ then $\langle g, h\rangle$ is abelian.

Proof. Note that the last part of the statement follows at once from Lemma 3.1. So we assume throughout the proof that $R_2 = \{0\}$. (i) Suppose $g, h \in N$. Assume that $gh \neq hg$. If $(gh)^2 \neq 1$ then, by Lemma 3.3 (i) it follows that $\langle g, h\rangle = \langle g, gh\rangle \cong G_{[16,3]}$.

So, to prove (i) we assume from now on that $(gh)^2 = 1$ and thus $gh = hg^{-1}$. Choose $x \in G \setminus N$. We may assume that $x^2 \neq 1$. Indeed, for otherwise, by Lemma 3.1, $gx = xg$ and thus $(gx)^2 = g^2x^2 = g^2 \neq 1$; so we can replace x by gx. We now claim that $\circ(g) = 4$ and therefore $\langle g, h\rangle \cong D_4$, as desired. We prove this by contradiction. Hence, assume $\circ(g) \neq 4$. Lemma 3.2 (ii) applied to the elements $g \in N$ and $x \in G \setminus N$ yields that either $gx = xg$ or $gx = x^{-1}g$.

First, assume that $gx = xg$. Then $(hx)^2 \neq 1$, because otherwise, by Lemma 3.1 (1), it follows that $ghx = hxg = hgx$, and hence $gh = hg$, a contradiction. Thus, because by assumption $\circ(g) \neq 4$, applying Lemma 3.2 (ii), to g and hx, we get that either $ghx = hxg = hgx$ or $ghx = (hx)^{-1}g$. The former is excluded as $gh \neq hg$. So $ghx = x^{-1}hg$. Hence, since $ghx = hg^{-1}x = hxg^{-1}$ and $x^{-1}hg = x^{-1}g^{-1}h = g^{-1}x^{-1}h$, we obtain that $(hx)^2 = g^2 = (x^{-1}h)^2$ and therefore $g^4 = (hx)^2(x^{-1}h)^2 = 1$. This gives a contradiction with the assumption $\circ(g) \neq 4$.

Second, assume that $gx = x^{-1}g$. Since $g^2 \neq 1$ we have that $(x^{-1}gx)^2 \neq 1$. So, applying Lemma 3.3 (i) to the elements g and $x^{-1}gx$ (recall that $\circ(g) \neq 4$), we get that g and $x^{-1}gx$ commute. So $gx^{-1}gx = x^{-1}gxg$ and thus $g^2x^2 = x^{-2}g^2$. Now, if $(hx)^2 = 1$ then, by Lemma 3.1 (1), it follows that $ghx = hxg$. Thus $hg^{-1}x = hgx^{-1}$ and hence $g^2 = x^2$. Then $g^2x^2 = x^{-2}g^2 = 1$. Hence $g^2 = x^2 = g^{-2}$ and therefore $\circ(g) = 4$, a contradiction. So $(hx)^2 \neq 1$ and we can apply Lemma 3.2 (ii) to g and hx. It follows that either $ghx = hxg$ or $ghx = (hx)^{-1}g$. We already have shown above that the former leads to a contradiction. Hence, $ghx = x^{-1}hg$. Since $ghx = hg^{-1}x = hx^{-1}g^{-1}$ and $x^{-1}hg = x^{-1}g^{-1}h = g^{-1}xh$, this yields that $xh(hx)^{-1} = g^2 = (x^{-1}h)^{-1}hx^{-1}$ and so $g^4 = (x^{-1}h)^{-1}hx^{-1}xh(hx)^{-1} = 1$. Hence $\circ(g) = 4$, again a contradiction. This finishes the proof of (i).

(ii) This follows at once from Lemma 3.1 (1).

(iii) Suppose that $h \in N$, $g \notin N$ and $gh \neq hg$. First assume that $(gh)^2 \neq 1$. Then we can apply Lemma 3.2 (iii) to g and gh. It follows that either $(g^\alpha(gh)^\beta)^2 = 1$ for all $\alpha, \beta \in \{-1, 1\}$, $ggh = g^2h = ghg^{-1}$ or $ggh = (gh)^{-1}g = h$. The latter is excluded as it yields $g^2 = 1$. The second possibility leads to $(gh)^2 = 1$ and is thus also excluded. It follows from Lemma 3.3 (iii) that $\langle g, h\rangle = \langle g, gh\rangle$ is isomorphic to either $G_{[16,3]}$ or $G_{[16,8]}$, as desired.

Second assume that $(gh)^2 = 1$. We claim that then $\circ(g) = 4$, and thus $\langle g, h\rangle \cong D_4$. Indeed, suppose the contrary, that is $g^4 \neq 1$. We then can apply part (ii) to g^2 and gh and we get $g^2gh = ghg^2 = gg^{-2}h$. Thus $g^4 = 1$, a contradiction.

(iv) Suppose $g, h \notin N$. If $(gh)^2 \neq 1$ then part (ii) yields that $\langle g, h\rangle = \langle h, gh\rangle$ is abelian. On the other hand, if $(gh)^2 = 1$ then part (iii) implies that $\langle g, h\rangle = \langle g, gh\rangle$ either is abelian or $\langle g, h\rangle \cong D_4$, because $(ggh)^2 = 1$ ∎

Lemma 3.5 *Assume that $(RG)^-_{\varphi_\sigma}$ is commutative. Let g and h be non-commuting elements of G. If $g^2 = h^2 = 1$ then $R_2 = \{0\}$ and $\langle g, h\rangle \cong D_4$.*

Proof. Assume that $gh \neq hg$. Hence, as $g^2 = h^2 = 1$, we get that $(gh)^2 \neq 1$. Since also $(g(gh))^2 = h^2 = 1$ and $\langle g, h\rangle = \langle g, gh\rangle$, the result follows from Lemma 3.4. ∎

Remark 3.6 *Lemmas 3.3, 3.4 and 3.5 imply that if G is a group of exponent 4 and $(RG)^-_{\varphi_\sigma}$ is commutative for a nontrivial orientation morphism then $g^2 \in \mathcal{Z}(G)$, for all $g \in G$. Thus $G' \subseteq \mathcal{Z}(G)$.*

We end this section by showing that if G is a nonabelian group with $(RG)^-_{\varphi_\sigma}$ commutative then G is a 2-group of exponent at most 8.

Proposition 3.7 *Let R be a commutative ring with $\text{char}(R) \neq 2$ and let G be nonabelian group. If $(RG)^-_{\varphi_\sigma}$ is commutative then G is a 2-group and its exponent is bounded by 8.*

Proof. It follows from Lemmas 3.3, 3.4 and 3.5 that noncentral elements g of G have order a divisor of 8. If y is a central element of G then yg is a noncentral element and thus $1 = (yg)^8 = y^8 g^8 = y^8$. Hence $o(y)$ divides 8 as well and the result follows. ∎

4 Groups of exponent eight

We know from Proposition 3.7 that if $(RG)^-_{\varphi_\sigma}$ is commutative then G is a 2-group of exponent bounded by 8. In this chapter we give a complete answer in case the exponent is precisely 8.

Theorem 4.1 *Let R be a commutative ring with $\text{char}(R) \neq 2$ and let G be a nonabelian group of exponent 8 with a nontrivial orientation homomorphism σ. Then, $(RG)^-_{\varphi_\sigma}$ is commutative if and only if one of the following conditions holds.*

(i) $R_2 = \{0\}$, $G = \langle g, h \mid g^8 = 1,\ h^2 = g^4,\ gh = hg^3 \rangle \times E$ *and* $N = \langle g^2, gh \rangle \times E$, *for some elementary abelian 2-group E.*

(ii) $\text{char}(R) = 4$, $G = \langle g, h \mid g^8 = 1,\ h^2 = g^4,\ gh = hg^{-1} \rangle \times E$ *and* $N = \langle g^2, h \rangle \times E$ *or* $N = \langle g^2, gh \rangle \times E$, *for some elementary abelian 2-group E.*

Proof. Suppose $(RG)^-_{\varphi_\sigma}$ is commutative and G is a 2-group of exponent 8. Let $A = \langle a \in G \mid \circ(a) = 8 \rangle$. By assumption, A is nontrivial. Fix $a \in A$ with $\circ(a) = 8$. Because of Lemma 3.3, we know that A is an abelian group. We claim that the elements of order 8 of G belong to $G \setminus N$, and that $h^{-1}gh = g^3$ or g^{-1}, for all $g \in A$ and $h \in G \setminus A$.

Since A is an abelian group generated by elements of order 8, it is enough to prove the claim for $g = a$ and $h \in G \setminus A$. Since $\circ(h) \leq 4$ and $\circ(a) = 8$, note that $ah \neq ha$. Indeed, because otherwise $\circ(ah) = 8$ while $ah \notin A$, a contradiction. Lemma 3.3 and Lemma 3.4 then yield that $\langle a, h \rangle = G_{[16,8]}$ or $G_{[16,9]}$, and by Propositions 2.3 and 2.4 also that $a \notin N$. In particular we obtain that $h^{-1}ah = a^{-1}$ or a^3, as desired. This finishes the proof of the claim.

Now, we show that A has index 2 in G. In order to show this, let $x, y \in G \setminus A$. Suppose that $xy^{-1} \in G \setminus A$. Then, by the previous paragraph, $a^{-2} = (xy^{-1})^{-1} a^2 (xy^{-1}) = y(x^{-1}a^2x)y^{-1} = ya^{-2}y^{-1} = a^2$. Hence, $a^4 = 1$, a contradiction. Therefore, $xy^{-1} \in A$, and thus indeed $[G : A] = 2$.

Next we show that A is the direct product of a cyclic group of order 8 and an elementary abelian 2-group. For this, first recall that every abelian group of finite exponent is a direct product of cyclic groups of prime power order (see for example [16, (5.1.2), p.92]). Because A is abelian of exponent 8, we thus get that A has a cyclic subgroup of order 8 as a direct factor. Without loss of generality, we may assume that this factor is $\langle a \rangle$. In order to show that A does not have a direct factor that is a cyclic group of order 4, it is sufficient to prove that $a^4 \in \{c^2, c^4\}$ for any element $c \in A$ with $c^2 \neq 1$. So, let $c \in A$ with $c^2 \neq 1$. Suppose that $a^4 \neq c^4$. Then $(ac)^4 = a^4 c^4 \neq 1$ and $\circ(ac) = 8$. So, by the claim above, $ac \in G \setminus N$. As $a \notin N$ and $[G : N] = 2$, we therefore obtain that $c \in N$. So, again by the above claim, $\circ(c) = 4$. Now, as in the beginning of the proof, let $h \in G \setminus A$. Then, $ah \neq ha$. As G is a 2-group and $\circ(h) \leq 4$, we may assume that $\circ(h) = 4$. Indeed, for otherwise, $h^2 = 1$ and by Lemma 3.1 it follows that $R_2 = \{0\}$. Then by Lemma 3.4 we have that $h \in N$,

$\circ(ah) = 4$ and $ah \in G \setminus A$. So, replacing h by ah we obtain the desired. Then, by Lemma 3.3, it follows that $R_2 = \{0\}$ and $\langle a, h\rangle$ is isomorphic to $G_{[16,8]}$, or $\mathrm{char}(R) = 4$ and $\langle a, h\rangle \cong G_{[16,9]}$. Consequently, $a^4 = h^2$. On the other hand, from the claim in the beginning of the proof we also know that $h^{-1}ch = c^{-1}$. Part (i) and (ii) of Lemma 3.2 then yield that either $ch = hc = c^{-1}h$, or $c^{-1}h = h^{-1}c = c^{-1}h^{-1}$, or $ch = h^{-1}c = c^{-1}h^{-1}$, or $c^2 = h^2$. Since $c^2 \neq 1$ and $h^2 \neq 1$, we deduce that $c^2 = h^2 = a^4$, as desired. So, $A = \langle a\rangle \times E$, with $E^2 = 1$.

Notice that by the first part of the proof $h^{-1}eh = e$ for all $e \in E$. Hence E is central in G. Hence, from Remark 2.11, G is the central product of $\langle a, h\rangle$ and E. Moreover, from the previous, either $R_2 = \{0\}$ and $\langle a, h\rangle \cong G_{[16,8]}$ or $\mathrm{char}(R) = 4$ and $\langle a, h\rangle \cong G_{[16,9]}$. Furthermore, as $\langle a\rangle \cap E = \{1\}$, either $ah = ha^3$ or $ah = ha^{-1}$, and $hA = G \setminus A$, we have that $\langle a, h\rangle \cap E = \{1\}$. Hence $G = \langle a, h\rangle \times E$.

To finish the proof of the necessity of the conditions, it remains to determine the kernels. By Remark 2.1 and Propositions 2.3 and 2.4 we get the desired kernels and also the sufficiency of the conditions follows. ∎

5 Groups of exponent four and abelian kernel

In the remainder of the paper we are left to deal with nonabelian 2-groups G of exponent 4. In this section we handle such groups for which the kernel N is abelian. Without specific reference to Remark 3.6 we will often use he fact that $g^2 \in Z(G)$ for $g \in G$ if $(RG)^-_{\varphi_\sigma}$ is commutative.

We first prove that if N is an elementary abelian 2-group then $(RG)^-_{\varphi_\sigma}$ is commutative.

Proposition 5.1 *Let G be a nonabelian group of exponent 4, R a commutative ring with $R_2 = \{0\}$ and σ a nontrivial orientation homomorphism. Assume that N is an elementary abelian 2-group. Then $(RG)^-_{\varphi_\sigma}$ is commutative.*

Proof. Since N is of index 2 and elementary abelian, the nonabelian group G contains an element x so that $G = N \cup xN$ and x has order 4. Furthermore, since $R_2 = \{0\}$, to prove the result, it is sufficient to show that $[g - \varphi_\sigma(g), h - \varphi_\sigma(h)] = 0$ for all $g, h \in G \setminus N$. Write $g = xa$ and $h = xb$ for some $a, b \in N$. Then

$$\begin{aligned}
[g - \varphi_\sigma(g), h - \varphi_\sigma(h)] &= [g + g^{-1}, h + h^{-1}] \\
&= [xa + ax^{-1}, xb + bx^{-1}] \\
&= xaxb + xabx^{-1} + ab + ax^{-1}bx^{-1} \\
&\quad -xbxa - xbax^{-1} - ba - bx^{-1}ax^{-1} \\
&= xaxb + ax^{-1}bx^{-1} + -xbxa - bx^{-1}ax^{-1}
\end{aligned}$$

Let $a', b' \in N$ so that $ax = xa'$ and $bx = xb'$. Since G has exponent 4, $x^2 \in N$ and N is abelian, we get that

$$[g + g^{-1}, h + h^{-1}] = x^2a'b + ab'x^{-2} - x^2b'a - ba'x^{-2} = 0,$$

as desired. ∎

Next, assume N is abelian but not an elementary abelian 2-group. The following lemma deals with elements of order 2 in N.

Lemma 5.2 *Let R be a commutative ring of $\mathrm{char}(R) \neq 2$, let G be a nonabelian group of exponent 4 and σ a nontrivial orientation homomorphism. Assume that $(RG)^-_{\varphi_\sigma}$ is commutative and N is abelian but not elementary abelian 2-group. Let $a \in N$. Then, $a^2 = 1$ if and only if $a \in \mathcal{Z}(G)$. Furthermore, $G_{[16,3]}$ is not a subgroup of G and if $x \in G \setminus N$ then x has order 4.*

Proof. Assume that $(RG)^-_{\varphi_\sigma}$ is commutative and N is abelian but not elementary abelian 2-group.

First, we show that if $x \in G \setminus N$ then $\circ(x) = 4$. Assume the contrary, that is assume $x \in G \setminus N$ and $x^2 = 1$. Then, by Lemma 3.4, we have that $ax = xa$, for all $a \in N$ with $a^2 \neq 1$. Because of the assumptions, N is generated by elements of order 4. Hence, we get that x is central and thus G is abelian, a contradiction.

Second, we show that $G_{[16,3]}$ is not a subgroup of G. Assume, the contrary. That is, suppose, that $H = G_{[16,3]} = \langle g, h \mid g^4 = h^4 = (gh)^2 = (gh^{-1})^2 = 1\rangle \subseteq G$. Clearly, $N \cap H$ is an abelian subgroup of index 2 in H. Since $\circ(gh) = \circ(gh^{-1}) = 2$, we know from the above that $gh \in N \cap H$. As N is abelian, $(g, h) \neq 1$ and thus $(g, gh) \neq 1$, $(h, gh) \neq 1$, we thus get that $g \notin N$ and $h \notin N$.

Take $a \in N$ with $a^2 \neq 1$. Because of Lemma 3.2 (ii), we get that either $ag = ga$, $ag = g^{-1}a$ or $a^2 = g^2$, and either $ah = ha$, $ah = h^{-1}a$ or $a^2 = h^2$. But $g^2 \neq h^2$. Also, since $gh \in N$, we have that $ag = ga$ is equivalent to $ah = ha$. Thus $ag = ga$ implies $ah \neq h^{-1}a$ and $a^2 \neq h^2$. Indeed, for otherwise we obtain that $h^2 = 1$ or $(ah)^2 = 1$. The former obviously is false. Because of the first part of the proof, the latter implies that $ah \in N$, again a contradiction. Similarly, $ah = ha$ implies that $ag \neq g^{-1}a$ and $a^2 \neq g^2$. So, we have to consider four remaining cases: $ag = ga$ and $ah = ha$, $ag = g^{-1}a$ and $ah = h^{-1}a$, $ag = g^{-1}a$ and $a^2 = h^2$, or $a^2 = g^2$ and $ah = h^{-1}a$. We show that each case leads to a contradiction.

Case 1: $ag = ga$ and $ah = ha$. Then, applying Lemma 3.2 (iii) to the elements g and ah, we have that $g(ah) = agh = ah^{-1}g^{-1}$ is equal to either $(ah)g$, $(ah)^{-1}g$, $(ah)g^{-1}$ or $(ah)^{-1}g^{-1}$. Thus either $gh = hg$, $g^2 = 1$, $h^2 = 1$ or $a^2 = 1$, a contradiction.

Case 2: $ag = g^{-1}a$ and $ah = h^{-1}a$. Since $gh^{-1} \in N$, we then have that $agh^{-1} = gh^{-1}a = gah = ag^{-1}h$. Hence $gh^{-1} = g^{-1}h$ and therefore $g^2 = h^2$, a contradiction.

Case 3: $ag = g^{-1}a$ and $a^2 = h^2$. Then, since $gh \in N$, we have that $gha = agh = g^{-1}ah$ and thus $g^2 = g^{-2} = hah^{-1}a^{-1} = (ha)^2$. Hence, applying Lemma 3.2 (iii) to the elements g and ah we have that $g(ah) = ag^{-1}h = ah^{-1}g$ is equal to either $(ah)g$, $(ah)^{-1}g$, $(ah)g^{-1}$ or $(ah)^{-1}g^{-1}$. So, either $h^2 = 1$, or $ah^{-1} = h^{-1}a^{-1} = ha$ and then $g^2 = haha = a^2 = h^2$, or $h^{-1}g = hg^{-1} = gh^{-1}$ and hence $gh = hg$, or $g^2 = ha^{-1}h^{-1}a^{-1} = (ha)^2a^2 = g^2a^2$ and then $a^2 = 1$. Therefore, each of the possibilities yields a contradiction.

Case 4: $a^2 = g^2$ and $ah = h^{-1}a$. Similarly as in Case 3, applying Lemma 3.2 (iii) to the elements ag and h, we obtain a contradiction.

So, indeed we have shown that G does not have $G_{[16,3]}$ as a subgroup.

Now, assume $a \in N$ with $\circ(a) = 2$. If $a \notin \mathcal{Z}(G)$ then by Lemma 3.1 it follows that $R_2 = \{0\}$. Moreover, there exists $x \in G \setminus N$ such that $ax \neq xa$ and $x^2 \neq 1$. As G has exponent 4 and $G_{[16,3]} \not\subset G$, Lemma 3.4 (iii) yields that $\langle a, x\rangle \cong D_4$. Then $\circ(ax) = 2$ and thus, by the first part of the proof $ax \in N$. This of course is impossible. Hence, we have shown that elements of order 2 in N are central in G. It remains to show that the converse holds. We prove this by contradiction. So, assume $a \in \mathcal{Z}(G) \cap N$ and $a^2 \neq 1$. Since N is an abelian subgroup of exponent 4 and of index 2 in G, and because G is not abelian, there exists $b \in G \setminus N$ and $c \in N$ such that $bc \neq cb$ and $\circ(c) = 4$. Again by the first part of the proof, $\circ(b) = 4$. Note also that $\circ(ba) = 4$. By Lemma 3.2 (ii), we have that $cb = b^{-1}c$ or $c^2 = b^2$. Assume first that $cb = b^{-1}c$. Then, applying Lemma 3.2 (ii) to the elements ba and c, we get that $b^2a^2 = c^2$. Hence $a^2 \neq c^2$, as $b^2 \neq 1$. Consequently, $(ac)^2 \neq 1$. Thus, applying Lemma 3.2 (ii) to the elements ba and ca, we obtain that $caba = b^{-1}a^{-1}ca = b^{-1}c$ or $b^2a^2 = c^2a^2$. Because $a^2 \neq 1$, the former is excluded. However, because $b^2a^2 = c^2$, the latter also yields that $a^2 = 1$, again a contradiction. Hence, $b^2 = c^2$. Notice that, if $a^2 = c^2 = b^2$ then $(ba)^2 = 1$ and, by Lemma 3.4 (ii), we thus have that $bac = cab$ and therefore $bc = cb$, a contradiction. So, $(ac)^2 \neq 1$. Lemma 3.2 (ii), applied to the elements b and ac, then yields that $acb = b^{-1}ac$ or $b^2a^2 = c^2$. However the latter is impossible as $a^2 \neq 1$ and $b^2 = c^2$. Therefore

$c^2 = b^2$ and $cb = b^{-1}c$. Now, applying Lemma 3.2 (ii) to the elements ba and c, we obtain that $cba = a^{-1}b^{-1}c$ or $b^2a^2 = c^2$. Both cases imply that $a^2 = 1$, a contradiction. This finishes the proof of the Lemma. ∎

We are now in a position the prove a solution to the problem in case the kernel is abelian.

Theorem 5.3 *Let R be a commutative ring with $\mathrm{char}(R) \neq 2$ and let G be a nonabelian group of exponent 4 with a nontrivial orientation homomorphism σ. Assume that N is abelian. Then, $(RG)^-_{\varphi_\sigma}$ is commutative if and only if one of the following conditions holds.*

- (i) *$R_2 = \{0\}$ and N is an elementary abelian 2-group.*
- (ii) *$G \cong Q_8 \times E$ and $N = C_4 \times E$, where C_4 is a cyclic group of order 4 and E is an elementary abelian 2-group.*
- (iii) *$G = \langle a, b \mid a^4 = b^4 = 1, ab = b^{-1}a\rangle \times E$ and $N = \langle a, b^2\rangle \times E$ or $N = \langle ab, b^2\rangle \times E$, where E is an elementary abelian 2-group.*

Proof. Assume $(RG)^-_{\varphi_\sigma}$ is commutative. Since G is not abelian, if N is elementary abelian 2-group then $R_2 = \{0\}$, by Lemma 3.1. So, suppose that N is not an elementary abelian 2-group. We need to show that either (ii) or (iii) holds.

First, suppose that $G_{[16,4]} \not\subseteq G$. Then, by Lemma 3.1, Lemma 3.3, Lemma 3.4 and Lemma 5.2, we have that G is a Hamiltonian 2-group, that is, $G \cong Q_8 \times E$, where $E^2 = 1$. Because N has index 2, it is then also clear that $N = C_4 \times E$ for some elementary abelian 2-subgroup E of G.

Second, suppose that $G_{[16,4]} = \langle g, h \mid g^4 = h^4 = 1, gh = h^{-1}g\rangle \subseteq G$. Then, since N is abelian, we have that $g \in G \setminus N$ or $h \in G \setminus N$. We claim that $h \in G \setminus N$. Indeed, for suppose the contrary. Then $h \in N$. Hence, by Lemma 3.2 (ii), we have that $hg = g^{-1}h$. Since $gh = h^{-1}g$, one deduces that $g^2 = h^2$, a contradiction. This finishes the proof of the claim.

Put $a = g$ if $g \notin N$, otherwise put $a = gh$. Clearly, $a \notin N$, $a^2 = g^2$, $ah = h^{-1}a$, $\circ(a) = 4$ and $\langle a, h\rangle = \langle g, h\rangle$. So N is an abelian group of exponent 4 and it contains ah. We claim that $N = \langle ah\rangle \times E$ for some elementary abelian 2-group E. For this it is sufficient to show that if $c \in N$ with $c^2 \neq 1$ then $\langle ah\rangle \cap \langle c\rangle \neq \{1\}$. Suppose the contrary. Then ah, c and ahc have order 4. Hence, because of Lemma 5.2, ah, c and ahc are not central in G. Because N has index 2 in G and since $h \notin N$, we get that $ch \neq hc$ and $ahch \neq hahc$. Hence, applying Lemma 3.2 (ii) to the elements ac and h, we get that $ahch = h^{-1}ahc$ or $(ahc)^2 = h^2$. Because $ah = h^{-1}a$, the former is excluded. Hence, $h^2 = (ahc)^2 = (ah)^2c^2 = a^2c^2$. Applying Lemma 3.2 (ii) to the elements c and h, we also obtain that $ch = h^{-1}c$ or $c^2 = h^2$. As $a^2 \neq 1$ and $h^2 = a^2c^2$, it follows that $ch = h^{-1}c$ must hold. So, $ahch = ahh^{-1}c = ac = hh^{-1}ac = hahc$ and thus $ahc \in \mathcal{Z}(G)$. Since also $ahc \in N$, Lemma 5.2 yields that $(ahc)^2 = 1$, in contradiction with $\circ(ahc) = 4$. This finishes the proof of the claim that $N = \langle ah\rangle \times E$ for some elementary abelian 2-group E. Again by Lemma 5.2, we also know that $E \subseteq \mathcal{Z}(G)$.

Because $(ah)^2 = a^2 = g^2 \neq h^2$, it is clear that $N = \langle ah\rangle \times \langle h^2\rangle \times E_0$, for some elementary abelian subgroup E_0 of E. Note that $\langle ah, h^2\rangle$ equals either $\langle g, h^2\rangle$ or $\langle gh, h^2\rangle$. Moreover, since the only central elements of order 2 in $\langle a, h\rangle = \langle g, h\rangle$ are g^2, h^2 and g^2h^2 and since none of these belong to E, we also get that $G = \langle g, h\rangle \times E_0$. This finishes the proof of the necessity of the conditions.

The sufficiency of the conditions follows from Remark 2.1, Proposition 2.2, Proposition 2.6 and Proposition 5.1. ∎

6 Groups of exponent four and nonabelian kernel

In this section we handle the remaining case, that is, we consider groups G of exponent four and with nonabelian kernel N. We first solve our problem in case all elements of order 2 in N are central in N.

Lemma 6.1 *Let R be a commutative ring with $\mathrm{char}(R) \neq 2$ and let G be a nonabelian group of exponent 4. Assume that N is not abelian and that the elements of order 2 in N are central in N. If $(RG)^-_{\varphi_\sigma}$ is commutative and $x \in G$ with $x^2 = 1$ then $x \in \mathcal{Z}(G)$. Furthermore, $\mathrm{char}(R) = 4$, N is a Hamiltonian 2-group and $G_{[16,3]}$ is not a subgroup of G.*

Proof. Assume $(RG)^-_{\varphi_\sigma}$ is commutative. If $R_2 \neq \{0\}$ then by Lemma 3.1 the first part of the result follows. So assume that $R_2 = \{0\}$ and fix h and h_1 in N so that $(h, h_1) \neq 1$. Because of the assumptions, $\circ(h) = \circ(h_1) = 4$. Let $x \in G$ with $\circ(x) = 2$. We need to show that $x \in \mathcal{Z}(G)$, or equivalently, $(x, g) = 1$ for all $g \in G$.

First, assume $x \notin N$. Then, by Lemma 3.4 (ii), for all $g \in N$ with $g^2 \neq 1$, we have that $gx = xg$. Note that, in particular, $hx = xh$. Now, consider $g \in N$ with $g^2 = 1$. Because, by assumption, g is central in N, we get that $(gh)^2 = g^2h^2 = h^2 \neq 1$. Hence, gh commutes with x and thus $xgh = ghx = gxh$. So, $xg = gx$. We thus have shown that $(x, g) = 1$ for all $g \in N$. Since N has index 2 in G and $x \notin N$, we get that $x \in \mathcal{Z}(G)$.

Second, assume that $x \in N$. By assumption, $x \in \mathcal{Z}(N)$. Let $g \in G \setminus N$. In order to prove that $x \in \mathcal{Z}(G)$, it is sufficient to show that $gx = xg$. If $g^2 = 1$ then this follows from the previous. If $(gx)^2 = 1$ or $(gh)^2 = 1$ then, again by the above, $gx, gh \in \mathcal{Z}(G)$. In the former case, $gx = xg$. In the latter case, $xgh = ghx = gxh$ and thus $xg = gx$, as desired. So, we may assume that $\circ(g) = \circ(gx) = \circ(gh) = 4$. By Lemma 3.2 (ii), for $y \in N$ with $\circ(y) = 4$, we then have three possibilities: (1) $gy = yg$, (2) $g^2 = y^2$ or (3) $yg = g^{-1}y$. Of course, this can be applied to the elements $y = h$ or $y = h_1$. It is therefore sufficient to consider the following three cases.

(1) $gh = hg$ (or, by symmetry, $gh_1 = h_1g$). Lemma 3.2 (ii), applied to the elements g and hx, yields that either $hxg = ghx = hgx$, $hxg = g^{-1}hx = hg^{-1}x$ or $g^2 = (hx)^2 = h^2$. So, either $xg = gx$, $(gx)^2 = gxgx = xg^{-1}gx = 1$ or $(gh)^2 = g^2h^2 = 1$. Because the latter two are excluded, we get that $xg = gx$, as desired.

(2) $g^2 = h^2$ and $gh \neq hg$ (or, by symmetry, $g^2 = h_1^2$ and $gh_1 \neq h_1g$). Because $xh = hx$, we have that $gxh \neq hgx$ and therefore, by Lemma 3.2 (ii), we get that either $hgx = xg^{-1}h$ or $(gx)^2 = h^2$. If $(gx)^2 = h^2 = g^2$ then $gx = xg$. Therefore, we may assume that $hgx = xg^{-1}h$. Lemma 3.2 (ii), applied the elements gx and hx, yields that either $hxgx = gxhx$ or $hxgx = xg^{-1}hx$ or $(gx)^2 = (hx)^2 = h^2$. In the first case we have that $hxg = gxh = ghx$. Hence, $gh = hxgx = xhgx = xxg^{-1}h = g^{-1}h$ and therefore $g = g^{-1}$, a contradiction. In the second case, we obtain $hxgx = xg^{-1}hx = hgxx = hg$ and hence $gx = xg$, as desired. In the third case, we have $(gx)^2 = h^2 = g^2$ and thus $gx = xg$, again as desired.

(3) $hg = g^{-1}h$, $gh \neq hg$, $g^2 \neq h^2$, $h_1g = g^{-1}h_1$, $gh_1 \neq h_1g$ and $gh_1 \neq h_1g$. Lemma 3.2 (ii), applied to the elements gx and h, gives us that either $gxh = hgx = g^{-1}hx$ (and hence $g = g^{-1}$, a contradiction), or $hgx = xg^{-1}h = xhg = hxg$ (and hence $gx = xg$) or $(gx)^2 = h^2$. So, we may assume that $(gx)^2 = h^2$. Similarly we that that $(gx)^2 = h_1^2$. Thus $h^2 = h_1^2$. So $hh_1 \in N$ has order 2 and thus is central in N. However, this is impossible as $(h, h_1) \neq 1$. This finishes the proof of the first part of the statement.

Since $G_{[16,3]}$ contains noncentral elements of order 2, it thus follows at once that $G_{[16,3]}$ is not a subgroup of G. Since N is not abelian and elements of order 2 are central, it hence follows, from Lemma 3.2 (i), that $\mathrm{char}(R) = 4$ and that every nonabelian subgroup of N generated by two

elements is isomorphic with Q_8. Hence, all subgroups of N are normal in N, i.e. N is a Hamiltonian 2-group. ∎

Theorem 6.2 *Let R be a commutative ring with $\mathrm{char}(R) \neq 2$ and let G be a nonabelian group of exponent 4 with a nontrivial orientation homomorphism σ. Assume that N is not abelian and that the elements of order 2 in N are central in N. Then $(RG)^-_{\varphi_\sigma}$ is commutative if and only if $\mathrm{char}(R) = 4$ and one of the following conditions holds.*

(i) G and N are Hamiltonian 2-groups.

(ii) $G = \langle a, b, c \mid a^4 = c^4 = 1,\ b^2 = a^2, ac = ca, ab = ba^{-1}, cb = bc^{-1}\rangle \times E$ and N is equal to either $\langle a, b\rangle \times \langle c^2\rangle \times E$ or $\langle a, cb\rangle \times \langle c^2\rangle \times E$.

Proof. Assume $(RG)^-_{\varphi_\sigma}$ is commutative. Because of Lemma 6.1 we know that the elements of order 2 in G are central, $\mathrm{char}(R) = 4$, $G_{[16,3]}$ is not a subgroup of G and N is a Hamiltonian 2-group, that is, $N = \langle a, b\rangle \times E$, where $\langle a, b\rangle = Q_8$ and $E^2 = 1$.

Clearly, if G contains an element c of order 2 that does not belong to N, then $G = Q_8 \times E \times \langle c\rangle$. Hence G also is a Hamiltonian 2-group and we are in case (i) of the theorem. So, to prove the necessity of the conditions, we may suppose that the elements in $G \setminus N$ have order 4. As E is central in G, Remark 2.11 yields that we then have that G is the central product of E and $\langle a, b, c\rangle$, where $c \in G \setminus N$. Hence $G = \langle a, b, c\rangle \times E_1$ for some subgroup E_1 of E. Replacing, if necessary, c by either ac or bc we may assume that c is not central in G. So $(a, c) \neq 1$ or $(b, c) \neq 1$.

Next we show that $\langle a, b, c\rangle = G_{[32,35]}$. By Lemma 3.2 *(ii)* we have that if $x \in \langle a, b\rangle$ with $\circ(x) = 4$ then one of following holds: $xc = cx$, $xc = c^{-1}x$ or $x^2 = c^2$.

Assume there exists $x \in \langle a, b\rangle$ with $\circ(x) = 4$ and $xc = cx$. Without loss of generality we may assume that $x = a$. So $ac = ca$ and thus $bc \neq cb$. Note that then $a^2 \neq c^2$ for otherwise ac is an element of order 2 not contained in N. It follows that $|\langle a, c\rangle| = 16$ and $\langle a, c\rangle = \langle a\rangle \times \langle c\rangle$. Clearly $\langle a, c\rangle \cap N = \langle a\rangle \times \langle c^2\rangle$ and, since $\circ(b) = \circ(ab) = \circ(a^{-1}b) = 4$, we thus get that $b \notin \langle a, c\rangle$. Hence $\langle a, b, c\rangle \geq 32$. Note that, by the above, $bc \neq cb$ and $b^2 \neq c^2$ imply that $bc = c^{-1}b$ then $\langle a, b, c\rangle = \langle a, b, c^2\rangle \cup \langle a, b, c^2\rangle c$ and $|\langle a, b, c^2\rangle| = 16$. So, $|\langle a, b, c\rangle| = 32$ and it is easily seen that $\langle a, b, c\rangle = G_{[32,35]}$.

If $ac = c^{-1}a$ and $bc = c^{-1}b$ then $(ab)c = c(ab)$. Because $\circ(ab) = 4$ the previous yields that $\langle a, b, c\rangle = G_{[32,35]}$.

If $ac = c^{-1}a$ and $a^2 = b^2 = c^2$ then $(bc)a = bac^{-1} = a^{-1}bc^{-1} = aa^2bc^2c = a(bc)$ and $b(bc) = c^3 = c^{-1}b^{-1}b = (bc)^{-1}b$. Note that $(bc)^2 \neq 1$ as $bc \notin N$. Hence bc is not central. So, replacing c by bc, we are again in a situation that c commutes with an element of order 4 in $\langle a, b\rangle$. Hence, $\langle a, b, c\rangle = G_{[32,35]}$. The case $bc = c^{-1}b$ and $a^2 = b^2 = c^2$ is dealt with similarly.

In order to finish the proof of the claim we now show that the following situation can not occur: $a^2 = b^2 = c^2$, $bc \neq cb$, $ac \neq ca$, $bc \neq c^{-1}b$ and $ac \neq c^{-1}a$. Lemma 3.2 *(iii)*, applied to the elements ac and bc, yields that either $(ac)(bc) = (bc)(ac)$, or $acbc = c^{-1}b^{-1}ac = cc^2b^2bac = ca^{-1}bc = c^{-1}abc$ (and hence $ac = c^{-1}a$, a contradiction) or $acbc = bcc^{-1}a = ba = ab^{-1}$ (and hence $cb = b^{-1}c^{-1} = bc$, a contradiction). Therefore $(ac)(bc) = (bc)(ac)$ (i.e., $acb = bca$) and thus all elements not in N commute. In particular, $(ac, acb) = 1$ and thus $(ac, b) = 1$. But then $bca = acb = bac$ and thus $ca = ac$, again a contradiction.

We already know that $G = \langle a, b, c\rangle \times E_1$ and $N = \langle a, b\rangle \times E$ with E_1 a subgroup of the elementary abelian 2-group E. Since $\langle a, b, c\rangle \cap N$ has index 2 in $\langle a, b, c\rangle$ and $c^2 \notin \langle a, b\rangle$ it follows that $E = \langle c^2\rangle \times E_1$. Hence, the necessity of the conditions follows from Proposition 2.7.

The proof of the sufficiency follows from Remark 2.1 and Propositions 2.2 and 2.7. ∎

Now it is only left to classify the groups G and the kernels N for which the φ_σ-antisymmetric elements commute in case N contains a noncentral element of order 2. Then, by Lemma 3.1, we have that $R_2 = \{0\}$. In order to proceed with this case we first prove the following lemma.

Assume $(RG)^-_{\varphi_\sigma}$ is commutative. Recall from Lemma 3.4 (i) that if x and y are noncommuting elements of N with $x^2 \neq 1$ and $y^2 = 1$ then $\langle x, y\rangle$ is either D_4 or $G_{[16,3]}$. In the next lemma we investigate the structure of the group $\langle x, y, z\rangle$ for $z \in G \setminus N$.

Lemma 6.3 *Assume $(RG)^-_{\varphi_\sigma}$ is commutative and G has exponent 4. Suppose $x, y \in N$ and $z \in G \setminus N$ are such that $x^2 \neq 1 = y^2$ and $z^2 \neq 1$. If $xy \neq yx$ then $R_2 = \{0\}$ and $\langle x, y\rangle$ is either D_4 or $G_{[16,3]}$. Moreover*

1. *If $\langle x, y\rangle = D_4$ then one of the following conditions holds.*
   - *(i) $yz = zy$ and $x^2 = z^2$;*
   - *(ii) $\langle y, z\rangle = D_4$, and $xz = z^{-1}x = zx^{-1}$;*
   - *(iii) $\langle y, z\rangle = G_{[16,3]}$, $xz = zx$ and $x^2 = (yz)^2$;*
   - *(iv) $\langle y, z\rangle = G_{[16,3]}$, $xz = z^{-1}x$ and $x^2 = z^2$;*
   - *(v) $\langle y, z\rangle = G_{[16,3]}$, $xz = z^{-1}x$ and $x^2 = (yz)^2$;*
   - *(vi) $\langle y, z\rangle = G_{[16,3]}$, $xz = z^{-1}x$ and $x^2 = (yz)^2z^2$;*
   - *(vii) $\langle y, z\rangle = G_{[16,3]}$, $xyz = zxy$ and $x^2 = z^2$.*
2. *If $\langle x, y\rangle = G_{[16,3]}$ then one of the following conditions holds.*
   - *(i) $yz = zy$, $xz = zx$ and $z^2 = x^2(xy)^2$;*
   - *(ii) $yz = zy$, $xz = z^{-1}x$ and $z^2 = x^2(xy)^2$;*
   - *(iii) $\langle y, z\rangle = G_{[16,3]}$, $xz = zx$, $xyz = zyx$ and $z^2 = (xy)^2$;*
   - *(iv) $\langle y, z\rangle = G_{[16,3]}$, $zxy = xyz = yzx$ and $x^2 = z^2$.*

Proof. Suppose $xy \neq yx$. As mentioned before the Lemma we already know that $R_2 = \{0\}$ and $\langle x, y\rangle$ is either D_4 or $G_{[16,3]}$. Since, by assumption, G has exponent 4, Lemma 3.4 (iii) yields that $\langle y, z\rangle$ is either abelian, D_4 or $G_{[16,3]}$. Because of Lemma 3.2 (ii), we also have that

$$xz = zx \quad xz = z^{-1}x \quad \text{ or } x^2 = z^2. \tag{2}$$

First, assume that $\langle x, y\rangle = D_4$. If $\langle y, z\rangle = D_4$ then, since $y^2 = 1$ and $z^2 \neq 1$, we get that $(yz)^2 = 1$. Therefore, Lemma 3.4 (ii), applied to x and yz, yields that $xyz = yzx$. Hence, $yx^{-1}z = yzx$ and thus $x^{-1}z = zx$. Because $x^2 \neq 1$, this implies in particular that $xz \neq zx$. Hence, (2) yields that $xz = z^{-1}z$ or $x^2 = z^2$. This with $x^{-1}z = zx$ implies that $zx^{-1} = xz = z^{-1}x$. Therefore, (ii) holds if $\langle y, z\rangle = D_4$. Assume now that $yz = zy$ (and thus $(yz)^2 \neq 1$). We claim that then $x^2 = z^2$, and thus (i) holds. Because of (2), we may assume that

$$xz = zx \quad \text{ or } \quad xz = z^{-1}x. \tag{3}$$

Lemma 3.2 (ii), applied to x and yz, yields that $xyz = yzx$, $xyz = z^{-1}yx$ or $x^2 = (yz)^2$. As $\langle x, y\rangle = D_4$ this implies that $x^{-1}z = zx$, $x^{-1}z = z^{-1}x$ or $x^2 = z^2$. Because $x^2 \neq 1$, we then obtain from (3) that $x^2 = z^2$.

Suppose now that $\langle y, z\rangle = G_{[16,3]}$. Lemma 3.2 (ii), applied to x and yz, yields that either $xyz = yzx$ (and hence $x^{-1}z = zx$), $xyz = z^{-1}x^{-1}y$ or $x^2 = (yz)^2$. Because of (2) we also know

that $xz = zx$, $xz = z^{-1}x$ or $x^2 = z^2$. If $xz = zx$ then it follows that $x^2 = (yz)^2$ and therefore (iii) holds. If $xz = z^{-1}x$ then we get that either $x^2 = z^2$, $(yz)^2z^2$ or $(yz)^2$, and therefore (iv), (v) or (vi) holds. If $x^2 = z^2$ then either $xz = z^{-1}x$ or $xyz = zxy$ and therefore (iv) holds or (vii) holds.

Second, assume that $\langle x, y\rangle = G_{[16,3]}$. First assume that $\langle y, z\rangle = D_4$. Then $(yz)^2 = 1$ and applying Lemma 3.4 (ii), to x and yz we have that $xyz = yzx$. Thus $xz \neq zx$. Since $(xy)^2 \neq 1$, Lemma 3.4, applied to xy and yz, yields that $xyyz = yzxz$. By Remark 3.6 the element y^2 is central. Hence it follows that $xzy = yzx$. This on its turn implies that $xzy = z^{-1}yx$ and therefore $xz \neq z^{-1}x$. From (2) we thus get that $x^2 = z^2$. Lemma 3.2 (ii), applied to xy and z, also yields us that $zxy = xyz = xz^{-1}y$ (and thus $zx = xz^{-1}$) , $z^{-1}xy = xyz = xz^{-1}y$ (and thus $z^{-1}x = xz^{-1}$) or $(xy)^2 = z^2 = x^2$ (and thus $yxy = x$). So each time we obtain a contradiction. So, $\langle y, z\rangle$ is not D_4.

Assume now that $yz = zy$. Then $(yz)^2 = z^2 \neq 1$ and applying Lemma 3.2 (ii), to x and yz, we have that either $xyz = yzx$, $xyz = yz^{-1}x$ or $x^2 = z^2$. Thus, since we know that either $xz = zx$, $xz = z^{-1}x$ or $x^2 = z^2$, it follows that $z^2 = x^2$ or $z^2 = x^2(xy)^2$. If $z^2 = x^2$ then, applying Lemma 3.2 (ii) to yx and z we get that either $yxz = zyx = yzx$, $yxz = yz^{-1}x$ or $(yx)^2 = x^2$. Since $xy \neq yx$ we thus obtain that $xz = zx$, or $xz = z^{-1}x$. Applying Lemma 3.2 (ii) to xy and yz, we get that either $xz = yzxy$, $xz = yz^{-1}xy$ or $(xy)^2 = x^2$. Again because $yx \neq xy$, we obtain that $z^2 = x^2(xy)^2$ and thus $(xy)^2 = 1$, a contradiction. Therefore $z^2 = x^2(xy)^2$ and hence (i) holds or (ii) holds.

Finally assume that $\langle y, z\rangle = G_{[16,3]}$. Lemma 3.2 (ii), applied to xy and z, yields that either $xyz = zxy$, $xyz = z^{-1}xy$ or $z^2 = (xy)^2$. If $xyz = zxy$ or $xyz = z^{-1}xy$ then, since $zy \neq yz$ and $yz \neq z^{-1}y$ and because of (2), we get that $x^2 = z^2$. If $z^2 = (xy)^2$ then, since $xy \neq yx$, we get that $z^2 \neq x^2$. Hence, (2) implies that $xz = zx$ or $xz = z^{-1}x$. We claim that then $xyz = zyx$. Suppose the contrary. Then, Lemma 3.2 (ii) applied to yx and yz, gives us that $yxyz = z^{-1}x$ or $(yz)^2 = (yx)^2 = (xy)^2 = z^2$. The former (together with $xz = zx$ or $xz = z^{-1}x$) implies that $yxyz = xz^{-1}$ or $yxyx = xz$. However this leads to a contradiction because it results in $z^2 = x^2(xy)^2 = x^2z^2$ and thus $x^2 = 1$. The latter gives a contradiction as it implies $yz = zy$.

So we are left deal with two cases: (Case 1) $xyz = zyx$ and $z^2 = (xy)^2$, and (Case 2) $x^2 = z^2$, and $xyz = zxy$ or $xyz = z^{-1}xy$.

(Case 1): $xyz = zyx$ and $z^2 = (xy)^2$. We show that then $xz = zx$ and thus (iii) holds. To prove this, we apply Lemma 3.2 (ii) to x and yz. This yields that either $xyz = yzx$, $xyz = z^{-1}yx$ or $x^2 = (yz)^2$. If $xyz = zyx = yzx$ or $xyz = zyx = z^{-1}yx$ then either $yz = zy$ or $(zy)^2 = 1$, a contradiction. Hence $x^2 = (yz)^2$ and thus $xz^{-1} = x^2xyzy$. Since $xyz = zyx$ this yields $xz^{-1} = x^2zyxy$. As x^2 and z^2 are central, we obtain that $xz^{-1} = zx(xy)^2 = zxz^2 = z^{-1}x$. Therefore $xz = zx$, as claimed.

(Case 2): $x^2 = z^2$, and $xyz = zxy$ or $xyz = z^{-1}xy$. We will prove that then $zxy = xyz = yzx$ and thus (iv) holds.

Clearly, $x^2 \neq (yz)^2$. Hence, it follows from Lemma 3.2 (ii) that $xyz = yzx$ or $xyz = z^{-1}yx$.

First assume that $xyz = yzx$. If $xyz = zxy$ then we are done. If, on the other hand, $xyz = z^{-1}xy$ then $yzx = xyz = z^{-1}xy$. Applying Lemma 3.2 (ii) to yx and yz, we get that either $yxyz = yzyx$, $yxyz = z^{-1}x$ or $(yx)^2 = (yz)^2$. Therefore we have that either $xyz = zyx$ and hence $yz = zy$ a contradiction, $zx = z^{-1}x$ and hence $z^2 = 1$, a contradiction or $xyx = zyz$ and hence $yz = z^{-1}xyx = xyzx = x^2yz$,obtaining that $x^2 = 1$, a contradiction.

Second assume that assume that $xyz = z^{-1}yx$. If $xyz = z^{-1}xy$ we have that $xy = yx$, a contradiction. Therefore to end the proof of the lemma we have to deal with the case that $x^2 = z^2$ and $zxy = xyz = z^{-1}yx$. Applying Lemma 3.2 (ii) to yx and yz, we get that either $yxyz = yzyx$, $yxyz = z^{-1}x$ or $(yx)^2 = (yz)^2$. In the first case we have that $xyz = zyx$ and since $xyz = zxy$ we have that $(x, y) = 1$, a contradiction. In the second case we have that $z^{-1}x = yxyz = yz^{-1}yx$ and hence $(z, y) = 1$, again a contradiction. Finally if $(yx)^2 = (yz)^2$ then $xyx = zyz$ and hence $z^{-1}xyx = yz$.

Since $zxy = xyz$ and $z^2 = x^2$ is central it follows that $yz = z^{-1}xyx = zxyx^{-1} = xyzx^{-1}$. Therefore we get that $xyz = yzx$ as desired. ∎

Theorem 6.4 *Let R be a commutative ring with $\mathrm{char}(R) \neq 2$ and let G be a nonabelian group of exponent 4 with a nontrivial orientation homomorphism σ. Assume that N is not abelian and that there exists a noncentral element of order 2 in N. Then $(RG)^-_{\varphi_\sigma}$ is commutative if and only if $R_2 = \{0\}$ and one the following conditions holds*

(*i*) $G \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab\rangle \times E$ *and* $N \cong D_4 \times E$*, where* $E^2 = 1$*;*

(*ii*) $G = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, ab = ba, ac = ca, ad = dab, bc = cb, bd = db, cd = da^2c\rangle \times E$ *and* $N = \langle b\rangle \times \langle c, d\rangle \times E$*, where* $E^2 = 1$*;*

(*iii*) $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, ab = ba, ac = ca^{-1}, bc = ca^2b^{-1}\rangle \times E$ *and* $N = \langle a, c\rangle \times \langle b^2\rangle \times E$*, where* $E^2 = 1$*.*

(*iv*) $G = \langle g, a, b \mid g^4 = a^4 = b^2 = 1,\ ga = ag,\ gb = bg,\ ab = g^2ba\rangle \times E$ *and* $N = \langle a, b\rangle \times E$ *or* $N = \langle ga, b\rangle \times E$*, where* $E^2 = 1$*.*

Proof. The sufficiency of the conditions follows from Remark 2.1, Proposition 2.5, Proposition 2.8, Proposition 2.9 and Proposition 2.10.

To prove the necessity, assume $(RG)^-_{\varphi_\sigma}$ is commutative. Let $N = \ker(\sigma)$. So N has index 2 in G. Since by assumption N contains a noncentral element of order 2, Lemma 3.1 yields that $R_2 = \{0\}$. Hence by Lemma 3.3 (*i*) we get that $Q_8 \not\subseteq N$.

Since N is not abelian, it then follows from [2] (see Theorem 3.3 and the introduction of Section 4) that either

(Case 1) The elements of order 4 in N generate an abelian subgroup.

(Case 2) N contains an elementary abelian 2-group of index 2.

(Case 1) Assume $A = \langle x \in N \mid x^2 \neq 1\rangle$ is an abelian subgroup. It follows from [13] that A is a subgroup of index 2 in N. Thus, write $G = N \cup Nh$ and $N = A \cup Ay$ for some $h \in G$ and $y \in N$ with $y^2 = 1$. Since $G_{[16,3]}$ is a nonabelian group generated by elements of order 4, it follows that it is not contained in N. Furthermore, as N is not abelian, Lemma 3.4 (*i*) yields that for $a \in A$, either $(a, y) = 1$, or $\langle a, y\rangle \cong D_4$ and $(ay)^2 = 1$. Hence, we can choose $x \in A$, with $x^2 \neq 1$ and so that $\langle x, y\rangle = D_4$. We also note that we may assume that $h^2 \neq 1$. Indeed, suppose $h^2 = 1$. Then, again Lemma 3.4 (*ii*), $(h, x) = 1$ and thus $(hx)^2 = h^2x^2 = x^2 \neq 1$. So, replacing h by hx if needed, we indeed may assume that $h^2 \neq 1$. Hence, by Lemma 3.4 (iii), $\langle y, h\rangle = D_4$ and thus $yh = h^{-1}y$.

Let $g \in N$ with $g^2 \neq 1$ (thus $g \in A$). Since elements of $N \setminus A$ have order 2, we get that $(y, g) \neq 1$. Furthermore, by the above, $\langle y, g\rangle = D_4$ and thus $yg = g^{-1}y$. By Lemma 3.2 (*ii*) we have that $hg = gh$, $gh = h^{-1}g$ or $h^2 = g^2$. We claim that

$$g^2 = h^2 \quad g^2 = (yh)^2 \quad \text{ or } g^2 = h^2(yh)^2. \tag{4}$$

First, assume that $hg = gh$. Then $(yh, g) \neq 1$ and, by Lemma 3.4 (*ii*), we have that $(yh)^2 \neq 1$. Therefore, Lemma 3.2 (*ii*), yields that either $gyh = h^{-1}yg$ or $(yh)^2 = g^2$ as desired. The latter is as desired. In the former case, $ghyh = hgyh = yg$ and thus $g^{-1}yhyh = g$. Hence $(yh)^2 = g^2$, again as desired in the claim. Second, assume that $gh = h^{-1}g$. Let $K = \langle yh, g\rangle$. If K is abelian then $yhg = gyh = g^2ygh = g^2yh^{-1}g = g^2yh^2hg = g^2h^2yhg$ and thus $h^2 = g^2$, again as desired. If K

is not abelian then, by Lemma 3.4 (*ii*), $(yh)^2 \neq 1$. Lemma 3.2 therefore yields that $g^2 = (yh)^2$ or $gyh = h^{-1}yg = h^{-1}g^{-1}y = g^{-1}hy$ and thus $g^2 = hyh^{-1}y^{-1} = (hy)^2h^2 = h^2(yh)^2$, again as desired. This proves the claim (4).

We now prove the following five statements.

(1.a): $\mathcal{Z}(N) = \{a \in A \mid a^2 = 1\}$ and $A/\mathcal{Z}(N)$ is an elementary abelian 2-group.

(1.b): $A/\mathcal{Z}(N)$ is cyclic, or equivalently, $A = \langle x \rangle \times E_1$ for some elementary abelian 2-group E_1.

(1.c): $\mathcal{Z}(N) \subseteq \mathcal{Z}(G)$.

(1.d): $G = \langle x, y, h \rangle \times E$ for some elementary abelian 2-subgroup E of G.

(1.e): $\langle x, y, h \rangle$ is isomorphic with either $G_{[32,30]}$, $G_{[16,13]}$ or $G_{[32,31]}$.

It then follows from Remark 2.1, Proposition 2.5, Proposition 2.8 and Proposition 2.9 that either condition (i), (ii) or (iii) of the statement of the result is satisfied. This then finishes the proof of (Case 1).

(1.a) First we show that if $1 \neq z \in \mathcal{Z}(N)$ then z has order 2. Indeed, for suppose z has order 4. Then $z \in A$. Since $y^2 = 1$ it follows that yz has order 4. Hence also $yz \in A$, a contradiction.

Since A is abelian, $N = A \cup Ay$ and $(y, x) \neq 1$, it follows that $\mathcal{Z}(N) \subseteq A$. Hence, $\mathcal{Z}(N) \subseteq \{a \in A \mid a^2 = 1\}$. Conversely, let $a \in A$ with $a^2 = 1$. If $(y, a) \neq 1$ then, by Lemma 3.5, $\langle y, a \rangle = D_4$. Hence, $(ya)^2 \neq 1$ and $ya \in N \setminus A$, a contradiction. So, $(y, a) = 1$ and thus $a \in \mathcal{Z}(N)$. So we have shown that $\mathcal{Z}(N) = \{a \in \mid a^2 = 1\}$.

Because of Remark 3.6, we also know that squares of elements of G are central. In particular, $A/\mathcal{Z}(N)$ is an elementary abelian 2-group.

(1.b) Because of part (1.a), in order to prove this property, it is sufficient to show that there does not exist an element $g \in A$ of order 4 so that $\langle g, x \rangle = \langle g \rangle \times \langle x \rangle$. Assume the contrary. By (4), we know that

$$x^2 = h^2, \;\; x^2 = (yh)^2 \text{ or } x^2 = h^2(yh)^2.$$

We will show that each of these cases leads to a contradiction. Note that, also by (4), $g^2 = h^2$, $g^2 = (yh)^2$ or $g^2 = h^2(yh)^2$.

Assume that $x^2 = h^2$. Since, by assumption $g^2 \neq x^2$, Lemma 3.2 (*ii*) yields that we $gh = hg$ or $gh = h^{-1}g$.

Suppose $gh = hg$. Since $(y, g) \neq 1$ (see above), we have that $(yh, g) \neq 1$ and therefore, by Lemma 3.4 (*ii*), $(yh)^2 \neq 1$. Applying Lemma 3.2 (*ii*) to yh and g, we deduce that $gyh = h^{-1}yg$ (and hence $g^2 = (yh)^2$) or $g^2 = (yh)^2$. So, $g^2 = (yh)^2$ and hence (as squares are central) $(y, h) = h^2(yh)^2 = h^2g^2$. Now, applying Lemma 3.2 (*ii*) to the elements yh and x, we get that $yhx = xyh$, $xyh = h^{-1}yx$ or $x^2 = (yh)^2$. We now show that each of these three cases leads to a contradiction. If $yhx = xyh = yx^{-1}h$ then $hx = x^{-1}h$ and hence $(x, h) = x^2 = h^2$. On the other hand, $(yxh)^2 = yxhyxh = (yh)^2x^2 = x^2g^2 \neq 1$, because of the assumption. Since also $(g, yhx) \neq 1$, Lemma 3.2 (*ii*) therefore implies that $gyxh = h^{-1}x^{-1}yg = hxyg = g^{-1}hxy$. Hence, $g^2 = (yxh)^2 = x^2g^2$ and thus $x^2 = 1$, a contradiction. If $xyh = h^{-1}yx = h^{-1}x^{-1}y = hxy$, then since $xyh = x(y, h)hy = xy^2h^2(yh)^2hy = h^2g^2xhy$, we get that $h^2g^2xh = hx$. Therefore, $h^2g^2 = (h, x)$. We also know that $(h, x) = h^2x^2(hx)^2 = (hx)^2$. Thus $h^2g^2 = (hx)^2$. Then, consider the group $\langle gx, h \rangle$. By Lemma 3.2 (*ii*), we get that either $gxh = hgx = ghx$ (and hence $xh = hx$,

a contradiction), or $gxh = h^{-1}gx = gh^{-1}x$ (and hence $xh = h^{-1}x$; so that $h^2g^2 = (xh)^2 = x^2$ and thus, because $h^2 = x^2$, we get that $g^2 = 1$, a contradiction), or $(gx)^2 = h^2$ (and hence$g^2x^2 = h^2$ and therefore $g^2 = 1$, a contradiction). So $xyh = h^{-1}yx$ is excluded. If $x^2 = (yh)^2$, then, since $(yh)^2 = g^2$, we get $x^2 = g^2$, again a contradiction. This shows that if $x^2 = h^2$ then $gh \neq hg$.

Therefore $x^2 = h^2$ implies that $gh = h^{-1}g$. Notice that $(h, x) \neq 1$ because otherwise $(hx)^2 = 1$ and hence, by Lemma 3.4 (*ii*), $ghx = hxg = hgx$. Then $hg = gh = h^{-1}g$ and hence $h = h^{-1}$, a contradiction. So, $(gh, x) \neq 1$ and, since $(gh)^2 = g^2 \neq x^2$, applying Lemma 3.2 (*ii*) to gh and x, one deduces that $xgh = h^{-1}g^{-1}x = h^{-1}xg^{-1}$. Since $xgh = xh^{-1}g$ and $h^2 = x^2$, we get that $g^2 = hx^{-1}h^{-1}x = (hx)^2$ and thus $(x, h) = (hx)^2 = g^2$. It follows that $(gx, h) = g^2h^2 \neq 1$, because otherwise $g^2 = h^2 = x^2$, a contradiction. Therefore, by Lemma 3.2 (*ii*), $gxh = h^{-1}gx$ or $(gx)^2 = h^2$. The former implies that $g^2h^2hgx = h^{-1}gx$ and thus $g^2 = 1$, a contradiction. The latter yields that $g^2x^2 = (gx)^2 = h^2 = x^2$ and thus $g^2 = 1$, again a contradiction.

So we have shown that indeed $x^2 \neq h^2$. Since x and g play a symmetric role and $x^2 \neq g^2$, we also get that $h^2 \neq g^2$ and it only remains to deal with the case that $x^2 = (yh)^2$ and $g^2 = h^2(yh)^2$ (and thus $(yh)^2 = x^2 = g^2h^2$). Again by Lemma 3.2 (*ii*), we have that $gh = hg$ or $gh = h^{-1}g$. Assume first that $(g, h) = 1$. Notice that then $(yh, g) \neq 1$ and hence, by Lemma 3.4 (*ii*), $(yh)^2 \neq 1$. Then, by Lemma 3.2 (*ii*) applied to yh and g, we get that $g^2 = (yh)^2$ or $gyh = h^{-1}yg = g^{-1}h^{-1}y$. In both cases this implies $g^2 = (yh)^2$. Then $h^2(yh)^2 = g^2 = (yh)^2$ and therefore $h^2 = 1$, a contradiction. Thus, $gh = h^{-1}g$. If $(hg, x) \neq 1$ then, by Lemma 3.2 (*ii*), we have that $g^2 = (hg)^2 = x^2$, a contradiction, or $xhg = g^{-1}h^{-1}x = g^2h^2ghx = g^2h^2h^{-1}gx = g^2hxg$. So, $g^2 = (h, x)$. Hence, $g^2 = (h, x) = h^2x^2(hx)^2 = g^2(hx)^2$ and thus $(hx)^2 = 1$. Then, by Lemma 3.4 (*ii*), we have that $hxg = ghx = h^{-1}gx = h^{-1}xg$ and thus $h^2 = 1$, a contradiction. Hence, $(hg, x) = 1$. Then $(h, x) = 1$. Notice that $(hx)^2 \neq 1$. Indeed, for otherwise, by Lemma 3.4 (*ii*), $(hx, x) = 1$ and thus $(h, x) = 1$, a contradiction. So, $1 = (hx)^2 = h^2x^2$ and thus $h^2 = x^2$, a contradiction. Also $(hg)^2 = g^2 \neq 1$. Applying Lemma 3.2 (*iii*) to hx and hg, we get that $hxhg = hghx$ (and hence $hg = gh$, a contradiction), or $hxhg = g^{-1}h^{-1}hx$ (and thus $h^2 = g^2$, a contradiction), or $hxhg = hgx^{-1}h^{-1}$ (and hence $x = x^{-1}$, a contradiction) or $1 = hxhghxhg = x^2h^2h^2g^2$ (and hence $x^2 = g^2$ again a contradiction).

So this finishes the proof of (1.b).

(1.c) We prove that $\mathcal{Z}(N) \subseteq \mathcal{Z}(G)$. So, let $e \in \mathcal{Z}(N)$. By means of contradiction assume that $(h, e) \neq 1$. If $(he)^2 = 1$ then, by Lemma 3.4 (*ii*), we have that $hex = xhe$. As $ex = xe$, we thus get that $hx = xh$. Also, by Lemma 3.4 (*ii*), we get that $xehe = hexe$. Thus, $(xe)(he) = hxe^2 = hx = xh = (xe)(eh)$ and hence $he = eh$, a contradiction. So, we also may assume that $(he)^2 \neq 1$. By Lemma 3.2 (ii), we have that $hx = xh$, $xh = h^{-1}x$ or $x^2 = h^2$. We now prove that each case leads to a contradiction.

First, assume $hx = xh$. Then, $(h, xe) \neq 1$ and hence, by Lemma 3.2 (ii), we have that either $xeh = h^{-1}xe = xh^{-1}e$ or $h^2 = x^2$. The former yields $(he)^2 = 1$, a contradiction. The latter implies $(hx)^2 = 1$ and thus, by Lemma 3.4 (*ii*), we get that $hxxe = xehx = ehx^2$. So, $he = eh$, a contradiction.

Second, assume $xh = h^{-1}x$. Applying Lemma 3.2 (*ii*) to h and xe, we get that $hxe = xeh = exh = eh^{-1}x$, or $xeh = h^{-1}xe = xhe$ or $h^2 = x^2$. The former leads to $(he)^2 = 1$, a contradiction. The second implies $he = eh$, a contradiction. So, $h^2 = x^2$. Since $(x, h) \neq 1$, we obtain that $(he, x) \neq 1$. Then, applying Lemma 3.2 (*ii*) to he and x, we get that $xhe = eh^{-1}x = exh = xeh$ and hence $he = eh$, a contradiction, or $(he)^2 = x^2 = h^2$ and hence $eh = he$, a contradiction.

Third, assume $x^2 = h^2$, $(h, x) \neq 1$ and $xh \neq h^{-1}x$. If $hexe = xehe$ then $hxe = xeh = x^3eh^3 = x^{-1}eh^{-1} = ex^{-1}h^{-1}$ and hence $(hxe)^2 = 1$. Then, by Lemma 3.4 (*ii*), we have that $(hxe, x) = 1$

and hence $(h,x)=1$, a contradiction. So, $(he,xe)\neq 1$. Therefore, applying Lemma 3.2 (*ii*) to he and xe, we get that $xehe=eh^{-1}xe$ and hence $xh=h^{-1}x$, a contradiction, or $(he)^2=x^2=h^2$ and hence $eh=he$, again a contradiction. This finishes the proof of (1.c).

(1.d) This follows at once from (1.b), (1.c) and Remark 2.11.

(1.e) We determine the group $\langle x,y,h\rangle$. Recall that either $x^2=h^2$, $x^2=(yh)^2$ or $x^2=(yh)^2h^2$ (see (4)). Also, remember that $\langle x,y\rangle=D_4$, $y^2=1$, $\circ(h)=4$, and thus, because of Lemma 3.4 (*iii*), either (1.e.i) $yh=hy$, (1.e.ii) $yh=h^{-1}y$ or (1.e.iii) $\langle y,h\rangle=G_{[16,3]}$. Note that $|\langle x,y,h\rangle|\leq 32$. We will deal with each of the three cases separately.

(1.e.i) Suppose $yh=hy$. Then $(yh)^2=h^2$. Since $x^2=h^2$, $x^2=(yh)^2$, or $x^2=(yh)^2h^2$, we thus get that $x^2=h^2$. Hence, $(h,x)=h^2x^2(hx)^2=(hx)^2$ and $(h,x)=(x,h)=x^2h^2(xh)^2$. So, $(xh)^2=(xh)^{-2}=(h^{-1}x^{-1})^2=(hx)^2$.

If $(xh)^2=1$ then $\langle x,y,h\rangle=\langle xy,y,xh\rangle$. Since $\circ(xy)=\circ(y)=\circ(xh)=o(c)$ and $(xy)y(xh)=y(xh)(xy)=(xh)(xy)y=x^2h$, we obtain that $\langle x,y,h\rangle=G_{[16,13]}$, as desired.

If $(xh)^2\neq 1$ then let $a=h$, $b=(xh)^2$, $c=y$ and $d=xy$. Clearly, $\circ(a)=4$, $\circ(b)=\circ(c)=\circ(d)=2$, $dab=xyh(xh)^2=yx^2(xh)^3=yx^2h^{-1}x^{-1}=yhx^{-1}=hxy=ad$, $da^2c=xyh^2y=y^2xh^2=y^2x^{-1}=yxy=cd$, $ab=ba$, $ac=ca$, $bc=cb$ and $bd=db$. It follows that $\langle x,y,h\rangle=G_{[32,30]}$, again as desired.

(1.e.ii) Suppose $yh=h^{-1}y$. Then, $(yh)^2=1$ and thus, by Lemma 3.4 (*ii*), we have that $xyh=yhx$. Hence, $xh=hx^{-1}$. Since, by Lemma 3.3 (*ii*), $xh=hx$, $xh=h^{-1}x$ or $x^2=h^2$, we have that $x^2=1$, $x^2=h^2$ or $xh=h^{-1}x$, respectively. Therefore if $yh=h^{-1}y$ then $xh=h^{-1}x$ and $x^2=h^2$. Then $\langle x,y,h\rangle=\langle xy,y,yh\rangle$ with $\circ(xy)=\circ(y)=\circ(yh)=2$ and $(xy)y(yh)=y(yh)(xy)=(yh)(xy)y=xyh$. Thus $\langle x,y,h\rangle=G_{[16,13]}$, as desired.

(1.e.iii) Suppose $\langle y,h\rangle=G_{[16,3]}=H\cup Hh$, where $H=\langle y\rangle\times\langle h^2\rangle\times\langle (yh)^2\rangle$ is an elementary abelian 2-group of order 8. Recall from Lemma 3.2 (*ii*) that $xh=hx$, $xh=h^{-1}x$ or $x^2=h^2$. We deal with each of these cases separately.

If $x^2=h^2$ then $|\langle x,y,h\rangle|=32$. Since $h^2\neq (yh)^2$, Lemma 3.2 (*ii*) yields that $x(yh)=(yh)x$ or $xyh=(yh)^{-1}x$, and thus $xh=h^{-1}x$ or $xyh=hxy$. If $xyh=hxy$ then $\langle x,y,h\rangle=G_{[32,30]}$. For this it is enough to note that $\langle x,y,h\rangle=\langle a,b,c\rangle$, with $a=h$, $b=(xh)^2$, $c=xy$ and $d=y$, and $\circ(a)=4$, $\circ(b)=\circ(c)=\circ(d)=2$, $dab=yh(xh)^2=yx^{-1}(xh)^{-1}=xyh^{-1}x^{-1}=h^{-1}xyx^{-1}=hy=ad$, $da^2c=yh^2xy=yxyh^2=x=cd$, $ab=ba$, $ac=ca$, $bc=cb$ and $bd=db$. If $xh=h^{-1}x$ then $\langle x,y,h\rangle=G_{[32,31]}$. To see this it is enough to note that $\langle x,y,h\rangle=\langle a,b,c\rangle$, with $a=x$, $b=yh$, $c=y$, $\circ(b)=4$, $ab=xyh=yh^{-1}x^{-1}=ba$, $ac=xy=ca^{-1}$ and $ca^2b^{-1}=yx^2h^{-1}y=yhy=bc$.

Suppose now that $xh=h^{-1}x$ and $x^2\neq h^2$. From (4) we know that we have to consider three cases: $x^2=h^2$, $x^2=(yh)^2$ or $x^2=(yh)^2h^2$. The former of course is excluded. If $x^2=(yh)^2$ then $\langle x,y,h\rangle=\langle a,b,c,d\rangle$, with $a=xh$, $b=h^2$, $c=xy$, $d=y$, and $\circ(a)=4$, $\circ(b)=\circ(c)=\circ(d)=2$, $ad=xhy=h^{-1}yx^{-1}=yh(yh)^2x^{-1}=yhx=yxhh^2=dab$, $da^2c=y(xh)^2xy=yx^{-1}y=x=cd$, $ab=ba$, $ac=xhxy=x^2(yh)^{-1}=x^2(yh)^2yh=yh=xyxh=ca$, $bc=cb$ and $bd=h^2y=yh^2=db$; so $\langle x,y,h\rangle=G_{[32,30]}$. If $x^2=(yh)^2h^2$ then take $a=xh$, $b=h^2$, $c=y$ and $d=xy$. Because $\circ(a)=4$, $\circ(b)=\circ(c)=\circ(d)=2$, $ad=xhxy=x^2(yh)^{-1}=x^2(yh)^2yh=h^2yh=xyxhh^2=dab$, $da^2c=xyx^2y=yxy=cd$, $ab=ba$, $ac=xhy=(yh)^{-1}x^{-1}=yh(yh)^2h^2x=yhx^2x=yxh=ca$, $bc=cb$ and $bd=db$, it follows that $\langle x,y,h\rangle=G_{[32,30]}$.

Assume that $xh=hx$. If $x^2\neq (yh)^2$ then, by Lemma 3.2 (*ii*), we have that $x(yh)=(yh)x$ or $x(yh)=(yh)^{-1}x$. Since $xyh=yhx^{-1}$ we get that $x^2=1$ or $x^2=(yh)^2$, a contradiction. Thus, $x^2=(yh)^2$. Let $a=x$, $b=h$, $c=y$. Clearly, $ab=ba$, $ac=ca^{-1}$ and $ca^2b^{-1}=yx^2h^{-1}=y(yh)^2h^{-1}=hy=bc$. Hence, since $|\langle x,y,h\rangle|=32$, we obtain that $\langle x,y,h\rangle=G_{[32,31]}$.

This finishes the proof of (1.e) and hence also the proof of (Case 1).

(Case 2) Assume N contains an elementary abelian 2-subgroup B of index 2 and that the elements of order 4 in N do not generate an abelian subgroup. We claim that if $c \in N$ with $c^2 \neq 1$ and $a \in \setminus B$ with $a^2 = 1$ then $\langle a, c \rangle$ is either abelian or D_4. Indeed, assume that $(a, c) \neq 1$. Then by Lemma 3.4 then either $\langle a, c \rangle = D_4$ or $\langle a, c \rangle = G_{[16,3]}$. Since $c = ab$ for some $b \in B$ it follows that $\langle a, c \rangle = \langle a, b \rangle$, a contradiction, because $G_{[16,3]}$ can not be generated by two elements of order 2. This proves the claim.

Next we claim that $a^2 \neq 1$ for all $a \in N \setminus B$. Suppose the contrary, then by the previous claim we have that for all $c \in N$ with $c^2 \neq 1$, $\langle a, c \rangle$ is either abelian or D_4. By the assumptions there exist $b_1 a, b_2 a \in N$ both of order 4 and $b_1, b_2 \in B$ so that $(b1_a, b_2 a) \neq 1$. Since $(b_i, a) \neq 1$ $(i = 1, 2)$ it follows that $\langle a, b_i \rangle = D_4 = \langle b_i a, b_j \rangle$ $(j, i = 1, 2)$. Hence $b_1 a b_2 a = (a b_2)^{-1} b_1 a = b_2 a b_1 a$, a contradiction. This finishes the proof of the claim.

As a consequence of the previous claim we have that D_4 can not be a subgroup of N, because otherwise we can always find an element of order 2 in $N \setminus B$.

Since Q_8 is not contained in G, Lemma 3.3 yields that N contains $G_{[16,3]}$. So, let $c \in N$ with $c^2 \neq 1$ and $b \in B$ such that $\langle b, c \rangle = G_{[16,3]}$.

Assume there exists $g \in G \setminus N$ with $g^2 = 1$. Then by Lemma 3.4 (ii) it follows that $(g, c) = 1$ and $1 = (g, cb)$. Therefore $1 = (g, b)$ and hence $(gb)^2 = 1$. Again by Lemma 3.4 (ii) we have that $1 = (gb, c) = (c, b)$, a contradiction. So we have shown that $g^2 \neq 1$ for all $g \in G \setminus N$.

Further, also choose $a \in N \setminus B$. Then, because of the claims above and Lemma 3.4 (i), for all $b \in B$ with $(a, b) \neq 1$ it follows that
$$\langle a, b \rangle = G_{[16,3]}.$$

Now we are going to show that $N = \langle a, b \rangle \times E$ for some elementary abelian 2-subgroup E of $\mathcal{Z}(N)$. Let $g \in G \setminus N$, so $g^2 \neq 1$. First we deal with the case $g^2 = a^2$. If $b, b_1 \in B$ such that $(a, b) \neq 1 \neq (a, b_1)$, then by Lemma 6.3 (2) it follows that $(a, b) = (g, a) = (a, b_1)$. Therefore $(a, bb_1) = 1$ and it follows that $B = \langle b \rangle \times \langle a^2 \rangle \times E$, for some elementary abelian 2-subgroup E of $\mathcal{Z}(N)$. Thus $N = \langle a, b \rangle \times E$ as desired. Second we deal with the case $g^2 \neq a^2$. If $(g, a) \neq 1$, then by Lemma 6.3 (2) it follows that $(a, b) = g^2 = (a, b_1)$. Therefore $(a, bb_1) = 1$ and again as above we have that $N = \langle a, b \rangle \times E$ as desired. Finally if $(g, a) = 1$ then again by Lemma 6.3 (2) the commutators (a, b) and (a, b_1) are either g^2 or $(ga)^2$. In case $(a, b) = (a, b_1)$ arguing as before we obtain the desired conclusion. So assume that $(a, b) \neq (a, b_1)$. But then, again by Lemma 6.3 (2) we have that $(a, bb_1) = (a, b)(a, b_1) = g^2 (ga)^2$ is either g^2 or $(ga)^2$. This is in contradiction with the fact that the elements in $G \setminus N$ are all of order 4.

We now show that there exists $g \in G \setminus N$ such that
$$\langle g, a, b \rangle = \langle g, a, b \mid g^4 = a^4 = b^2 = 1,\ ga = ag,\ gb = bg,\ ab = g^2 ba \rangle = G_{[32,24]}.$$

For this note that by Lemma 3.2 (ii), for any $g \in G \setminus N$ we have either (1) $ag = ga$ or (2) $ag = g^{-1}a$ or (3) $a^2 = g^2$. First assume (1), that is $ag = ga$. Then it is easy to verify that case (2) (i) or case (2) (iii) of Lemma 6.3 must holds. In the first case it is readily verified that $\langle g, a, b \rangle = G_{[32,24]}$ and in the second case, replacing g by ga, one also obtains that $\langle g, a, b \rangle = \langle ga, a, b \rangle = G_{[32,24]}$. Assume now (2), that is $ag = g^{-1}a$. Then (2) (ii) of Lemma 6.3 holds. Thus $gb \in \mathcal{Z}(\langle g, a, b \rangle)$ and $(gb)^2 = (a, b)$. Therefore $\langle g, a, b \rangle = \langle gb, a, b \rangle = G_{[32,24]}$. Third assume (3), that is, $a^2 = g^2$. Then (2) (iv) of Lemma 6.3 holds. Thus $gab \in \mathcal{Z}(\langle g, a, b \rangle)$ and $(gab)^2 = (a, b)$. Therefore $\langle g, a, b \rangle = \langle gab, a, b \rangle = G_{[32,24]}$ as desired.

Now we are going to prove that $E \subseteq \mathcal{Z}(G)$ and therefore $G = G_{[32,24]} \times E$ finishing the proof of the theorem. Let $e \in E$. We need to show that $(g, e) = 1$. From the above we know that there exists $g \in G \setminus N$ such that $\langle g, a, b \rangle = \langle g, a, b \mid g^4 = a^4 = b^2 = 1,\ ga = ag,\ gb = bg,\ ab = g^2 ba \rangle$. Applying Lemma 3.2 (ii) to ae and g we have that either $aeg = gae$ and thus $(g, e) = 1$ as desired;

or $aeg = g^{-1}ae$ then $eg = g^{-1}e$ and thus $(ge)^2 = 1$, a contradiction because $ge \notin N$; or $g^2 = (ae)^2$ and thus $g^2 = a^2$, a contradiction. ∎

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